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On Different Results of New Multistep Iteration Method for Integral Type Mapping in Banach Spaces

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Abstract. In this paper, we prove that the new multistep iteration process converges to unique fixed points of integral type mappings. Also, we show that some stability results for this multistep iteration method by using integral type mapping in Banach spaces. Finally, new multistep iteration method is equivalent to Mann iteration method when applied to this mapping.

Keywords. Integral Type, Fixed Point, Stable.

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1. Introduction and Preliminaries

Throughout this paper, we denote that the set of natural numbers, real numbers and positive real numbers by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ respectively. Let X be a nonempty set and $T : X \to X$ be a self-map. The set of all fixed points of T is denoted by F_T.Fixed point theory is the one of the basic and the most widely applied different areas of mathematics. Iteration schemes and contraction type mappings are one of the most important in the theory. The first result on fixed points for contraction type mappings was well known Banach contraction theorem [3]. A generalization of the Banach contraction principle [3] was established by Branciari [7] as follows: Let (X,d) be a complete metric space, $c \in (0,1)$, and let $T : X \to X$ be a mapping such that for each $x, y \in X$,

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt \tag{1}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable mapping which is summable, nonnegative and such that for each $\varepsilon > 0$, $\int\limits_0^{\varepsilon} \varphi(t)dt > 0$. Then, T has a unique fixed point $x_0 \in X$ such that, for each $x \in X$, $\lim\limits_{n\to\infty} T^n x = x_0$.

Several researchers, who studied integral type inequalities, are results as contained in the reference section. We can see some of authors. ([1], [2], [6], [7], [9], [17], [18], [21], [22], [23], [24], [25], [30], [32]).

Many iteration schemes have been defined in the literature of fixed point theory. Now we give the iterations which will be used in the study.

Let X be a normed space, C be a nonempty convex subset of an arbitrary normed space X, and $T : C \to C$ be a self-mapping of C. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n^i\}_{n=0}^{\infty}$ are real sequences for all $n \in \mathbb{N}$.

The Mann iterative process [20] is as follows;

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n \end{cases} \tag{2}$$

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where $\{\alpha_n\}_{n=0}^{\infty} \in [0,1]$ and $\sum\limits_{n=0}^{\infty} \alpha_n < \infty$ for all $n \in \mathbb{N}$.

New multistep iterative scheme [11] is defined by,

$$\left\{ \begin{array}{c} f(T,x_n) = (1-\alpha_n)y_n^1 + \alpha_n y_n^1 \\ y_n^i = (1-\beta_n^i)y_n^{i+1} + \beta_n^i T y_n^{i+1},\ i = 1,2,...k-2 \\ y_n^{k-1} = (1-\beta_n^{k-1})x_n + \beta_n^{k-1} T x_n \end{array} \right. \tag{3}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1)$, $\sum\limits_{n=0}^{\infty} \alpha_n < \infty$ and $\left\{\beta_n^i\right\}_{n=0}^{\infty} \subset [0,1)$, $i = \overline{1,k-1}$ for all $n \in \mathbb{N}$.

To obtain our main results, we need some definitions and lemmas in the following.

The topic of stability results for various maps of fixed point theory has been studied by many researchers (see [1], [2], [5], [10], [13], [14], [15], [19], [23], [25], [26], [27], [28], [31]). We will use defination for the stability of iteration methods is given by Harder and Hicks [13, 14]

Definition 1. [13, 14] Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping of X. Suppose that $F_T = \{p \in X : Tp = p\}$ is the set of fixed points of T. Let $\{x_n\}_{n=0}^{\infty} \subset X$ be sequence generated by an iteration scheme defined by

$$x_{n+1} = f(T,x_n),\ n = 0,1,2... \tag{4}$$

where $x_0 \in X$ is the initial approximation and f is some function. Suppose $\{x_n\}_{n=0}^{\infty} \subset X$ converge to a fixed point of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and set $\varepsilon_n = d(y_n, f(T,y_n))$, $n = 0,1,..$ .Then the iteration scheme (4) is $T-$stable if and only if $lim_{n\to\infty}\varepsilon_n = 0 \Rightarrow lim_{n\to\infty} y_n = p$.

Let X be normed space and $\{y_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence, if Definition 1 derived according to new mutistep iteration method (3). Then we have,

$$\begin{aligned} f(T,y_n) &= (1-\alpha_n)z_n^1 + \alpha_n z_n^1 \\ z_n^i &= (1-\beta_n^i)z_n^{i+1} + \beta_n^i T z_n^{i+1},\ i = 1,2,...k-2 \\ z_n^{k-1} &= (1-\beta_n^{k-1})y_n + \beta_n^{k-1} T y_n \end{aligned}$$

and

$$\varepsilon_n = \|y_n - f(T,y_n)\|.$$

Definition 2. [18] The function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called subadditive integrable function if and only if for all a,b,c are a real number

$$a \le b + c \Rightarrow \int_0^a \varphi(t)dt \le \int_0^b \varphi(t)dt + \int_0^c \varphi(t)dt \tag{5}$$

Lemma 1. [33] Let $\{t_n\}_{n=0}^{\infty}$ and $\{k_n'\}_{n=0}^{\infty}$ be nonnegative real sequence satisfying the following inequality:

$$t_{n+1} \le (1-\lambda_n)t_n + k_n',\ n = 0,1,...$$

where $\lambda_n \in (0,1)$, for all $n \ge n_0$, $\sum\limits_{n=0}^{\infty} \lambda_n = \infty$ and $\frac{k_n'}{\lambda_n} \to 0$ as $n \to \infty$. Then $\lim\limits_{n\to\infty} t_n = 0$.

Lemma 2. [23] Let (X,d) be a complete metric space and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable mapping which is summable, nonnegative and such that for each $\varepsilon > 0$, $\int\limits_0^{\varepsilon} \varphi(t)dt > 0$. Suppose that $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty} \subset X$ and $\{a_n\}_{n=0}^{\infty} \subset (0,1)$ are sequences such that

$$\left| d(u_n, v_n) - \int_0^{d(u_n,v_n)} \varphi(t)dt \right| \le a_n,$$

with $\lim_{n\to\infty} a_n = 0$.

2. Mathematical formulas and theorems

Theorem 1. Let C be a nonempty closed convex subset of an arbitrary Banach space X and $T : C \to C$ be a self-mapping of C satisfying (1) with $F_T \neq \varnothing$. Let $\{x_n\}$ be a sequence defined by (3) with real sequence $\{\alpha_n\}_{n=0}^{\infty} \in (0,1)$ for all $n \in \mathbb{N}$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a Lebesgue integrable mapping which is summable, nonnegative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$. Then, the iterative sequence $\{x_n\}$ converges to the fixed point of T.

Proof. Let p be fixed point of T and $\theta \in (0,1)$. We will show that $\{x_n\}$ converges to the fixed point $p = Tp$, by using condition (3) and Lemma 2. We get

$$\begin{aligned}
\int_0^{\|x_{n+1}-p\|} \varphi(t)dt &\leq \left\|(1-\alpha_n)y_n^1 + \alpha_n T y_n^1 - p\right\| + a_n \\
&\leq (1-\alpha_n)\int_0^{\left\|y_n^1-p\right\|} \varphi(t)dt + \alpha_n \int_0^{\left\|Ty_n^1-p\right\|} \varphi(t)dt + 2a_n \\
&\leq (1-\alpha_n)\int_0^{\left\|y_n^1-p\right\|} \varphi(t)dt + \alpha_n c \int_0^{\left\|y_n^1-p\right\|} \varphi(t)dt + 2a_n \\
&\leq [1-\alpha_n(1-c)]\int_0^{\left\|y_n^1-p\right\|} \varphi(t)dt + 2a_n
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
\int_0^{\left\|y_n^1-p\right\|} \varphi(t)dt &\leq \left\|y_n^1 - p\right\| + a_n \\
&\leq (1-\beta_n^1)\int_0^{\left\|y_n^2-p\right\|} \varphi(t)dt + \beta_n^1 \int_0^{\left\|Ty_n^2-p\right\|} \varphi(t)dt + 2a_n \\
&\leq [1-\beta_n^1(1-c)]\int_0^{\left\|y_n^2-p\right\|} \varphi(t)dt + 2a_n
\end{aligned} \tag{7}$$

and, we have

$$\int_0^{\left\|y_n^2-p\right\|} \varphi(t)dt \leq [1-\beta_n^2(1-c)]\int_0^{\left\|y_n^3-p\right\|} \varphi(t)dt + 2a_n \tag{8}$$

and continuing the above process

$$\int_0^{\left\|y_n^{k-2}-p\right\|} \varphi(t)dt \leq [1-\beta_n^{k-2}(1-c)]\int_0^{\left\|y_n^{k-1}-p\right\|} \varphi(t)dt + 2a_n \tag{9}$$

and we have

$$\int_0^{\left\|y_n^{k-1}-p\right\|} \varphi(t)dt \leq [1-\beta_n^{k-1}(1-c)]\int_0^{\|x_n-p\|} \varphi(t)dt + 2a_n \tag{10}$$

substituting (7), (8), (9) and (10) in (6), we obtain

$$\begin{aligned}\int_0^{\|x_{n+1}-p\|}\varphi(t)dt &\leq [1-\alpha_n(1-c)]\int_0^{\|y_n^1-p\|}\varphi(t)dt+2a_n\\ &\leq [1-\alpha_n(1-c)][1-\beta_n^1(1-c)]\int_0^{\|y_n^2-p\|}\varphi(t)dt+2\cdot 2a_n\\ &\leq [1-\alpha_n(1-c)][1-\beta_n^1(1-c)][1-\beta_n^2(1-c)]\int_0^{\|y_n^3-p\|}\varphi(t)dt+3\cdot 2a_n\\ &\ \ \ \ \dots\\ &\leq [1-\alpha_n(1-c)][1-\beta_n^1(1-c)]...[1-\beta_n^{k-1}(1-c)]\int_0^{\|x_n-p\|}\varphi(t)dt+k\cdot 2a_n\end{aligned}$$

and

$$[1-\alpha_n(1-c)][1-\beta_n^1(1-c)]...[1-\beta_n^{k-1}(1-c)]\leq[1-\alpha_n(1-c)] \tag{11}$$

where $k=1,2,...,$ k is a finete and using inequality (11), we get

$$\int_0^{\|x_{n+1}-p\|}\varphi(t)dt\leq[1-\alpha_n(1-c)]\int_0^{\|x_n-p\|}\varphi(t)dt+k2a_n \tag{12}$$

from Lemma 2, we get $\lim_{n\to\infty} a_n=0$, $[1-\alpha_n(1-c)]<1$ and if we suppose that $\lambda_n=\alpha_n(1-c)$, $t_n=\int_0^{\|x_n-p\|}\varphi(t)dt$ and $k_n'=2ka_n$, then $\lim_{n\to\infty}\int_0^{\|x_n-p\|}\varphi(t)dt=0$. The proof is complete. ∎

Theorem 2. Let $(X,\|\cdot\|)$ be an arbitrary Banach space, $T:C\to C$ be a self-mapping of C satisfying (1) with $F_T\neq\varnothing$, and p the unique fixed point of T. Let $\{x_n\}$ be a sequence defined by (3) with real sequences $\{\alpha_n\},\{\beta_n^i\}\in(0,1)$ for all $n\in\mathbb{N}$. Let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ be a Lebesgue integrable mapping which is summable, nonnegative and such that for each $\varepsilon>0$, $\int_0^{\varepsilon}\varphi(t)dt>0$. Then, the iterative sequence $\{x_n\}$ is $T-$stable.

Proof. Suppose that $\{y_n\}_{n=0}^{\infty}\subset C$ be an arbitrary sequence and set $\varepsilon_n=\|y_n-f(T,y_n)\|$, $n=0,1,...$ and let $\lim_{n\to\infty}\varepsilon_n=0$. Then, we will show that $\lim_{n\to\infty}y_n=p$.

Using condition (1) and Lemma 2, we have

$$\begin{aligned}\int_0^{\|y_{n+1}-p\|}\varphi(t)dt &\leq \|y_{n+1}-p\|+a_n\\ &\leq \left\|y_{n+1}-(1-\alpha_n)z_n^1-\alpha_nTz_n^1+(1-\alpha_n)z_n^1+\alpha_nTz_n^1-p\right\|+a_n\\ &\leq \int_0^{\varepsilon_n}\varphi(t)dt+(1-\alpha_n)\int_0^{\|z_n^1-p\|}\varphi(t)dt+\alpha_n\int_0^{\|Tz_n^1-p\|}\varphi(t)dt+3a_n\\ &\leq \int_0^{\varepsilon_n}\varphi(t)dt+[1-\alpha_n(1-c)]\int_0^{\|z_n^1-p\|}\varphi(t)dt+3a_n\end{aligned} \tag{13}$$

By using condition (1), we get

$$\begin{aligned}\int_0^{\|z_n^1-p\|}\varphi(t)dt &\leq \left\|z_n^1-p\right\|+a_n\\ &\leq [1-\beta_n^1(1-c)]\int_0^{\|z_n^2-p\|}\varphi(t)dt+2a_n\end{aligned} \tag{14}$$

and

$$\begin{aligned}\int_0^{\|z_n^2-p\|}\varphi(t)dt &\leq \|z_n^2-p\|+a_n\\ &\leq [1-\beta_n^1(1-c)]\int_0^{\|z_n^3-p\|}\varphi(t)dt+2a_n \quad (15)\end{aligned}$$

and

$$\int_0^{\|z_n^{k-1}-p\|}\varphi(t)dt \leq [1-\beta_n^{k-1}(1-c)]\int_0^{\|y_n-p\|}\varphi(t)dt+2a_n \quad (16)$$

we combine above inequalities and replace them in (13). We get

$$\begin{aligned}\int_0^{\|y_{n+1}-p\|}\varphi(t)dt &\leq \int_0^{\varepsilon_n}\varphi(t)dt+[1-\alpha_n(1-c)]\int_0^{\|z_n^1-p\|}\varphi(t)dt+3a_n\\ &\leq \int_0^{\varepsilon_n}\varphi(t)dt+[1-\alpha_n(1-c)][1-\beta_n^1(1-c)]\int_0^{\|z_n^1-p\|}\varphi(t)dt+5a_n\\ &\leq \ldots\\ &\leq [1-\alpha_n(1-c)][1-\beta_n^1(1-c)]\ldots[1-\beta_n^{k-1}(1-c)]\int_0^{\|y_n-p\|}\varphi(t)dt+\\ &\quad +\int_0^{\varepsilon_n}\varphi(t)dt+(2k+1)a_n\end{aligned}$$

and using inequalities (11), we have

$$\int_0^{\|y_{n+1}-p\|}\varphi(t)dt \leq [1-\alpha_n(1-c)]\int_0^{\|y_n-p\|}\varphi(t)dt+\int_0^{\varepsilon_n}\varphi(t)dt+(2k+1)a_n \quad (17)$$

k is finete and an application of Lemma 1 to (17) $\lim_{n\to\infty} y_n = p$.

Conversely, assume that $\lim_{n\to\infty} y_n = p$. We prove that $\lim_{n\to\infty}\varepsilon_n = 0$. From condition (1) and using Lemma 2, we get

$$\begin{aligned}\int_0^{\varepsilon_n}\varphi(t)dt &= \int_0^{\|y_{n+1}-f(T,y_n)\|}\varphi(t)dt\\ &\leq (1-\alpha_n)\|z_n^1-p\|+\alpha_n\|Tz_n^1-p\|+\|y_{n+1}-p\|+a_n\\ &\leq (1-\alpha_n)\int_0^{\|z_n^1-p\|}\varphi(t)dt+\alpha_n\int_0^{\|Tz_n^1-p\|}\varphi(t)dt+\int_0^{\|y_{n+1}-p\|}\varphi(t)dt+3a_n\\ &\leq [1-\alpha_n(1-c)]\int_0^{\|z_n^1-p\|}\varphi(t)dt+\int_0^{\|y_{n+1}-p\|}\varphi(t)dt+3a_n\end{aligned}$$

By combining (14), (15), (16), and (11), we obtain

$$\begin{aligned}
\int_0^{\varepsilon_n}\varphi(t)dt &\leq [1-\alpha_n(1-c)]\int_0^{\left\|z_n^1-p\right\|}\varphi(t)dt+\int_0^{\|y_{n+1}-p\|}\varphi(t)dt+3a_n \\
&\leq [1-\alpha_n(1-c)][1-\beta_n^1(1-c)]\int_0^{\left\|z_n^2-p\right\|}\varphi(t)dt+\int_0^{\|y_{n+1}-p\|}\varphi(t)dt+5a_n \\
&\leq [1-\alpha_n(1-c)][1-\beta_n^1(1-c)]...[1-\beta_n^{k-1}(1-c)]\int_0^{\|y_n-p\|}\varphi(t)dt+ \\
&\quad +\int_0^{\|y_{n+1}-p\|}\varphi(t)dt+(2k+1)a_n \\
&\leq [1-\alpha_n(1-c)]\int_0^{\|y_n-p\|}\varphi(t)dt+\int_0^{\|y_{n+1}-p\|}\varphi(t)dt+(2k+1)a_n \qquad (18)
\end{aligned}$$

By taking $n\to\infty$ of (18) and using $\lim\limits_{n\to\infty} a_n=0$, $\lim\limits_{n\to\infty}\int_0^{\|y_n-p\|}\varphi(t)dt=0$ and $\lim\limits_{n\to\infty}\int_0^{\|y_{n+1}-p\|}\varphi(t)dt=0$. We have $\lim\limits_{n\to\infty}\varepsilon_n=0$. The proof is complete. ■

Theorem 3. Let C be a nonempty closed convex subset of an arbitrary Banach space X and $T:C\to C$ be a self-mapping of C satisfying (1) with $F_T\neq\varnothing$. Let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ be a Lebesgue integrable mapping which is summable, nonnegative and such that for each $\varepsilon>0$, $\int\limits_0^{\varepsilon}\varphi(t)dt>0$. If $u_0=x_0\in C$ then the following statements are equivalent.

(i) Mann iteration (2) converges to fixed point p;

(ii) New multistep iteration (3) converges to fixed point p.

Proof. We will prove $(i)\Rightarrow(ii)$, let new multistep iteration method (3) converge to p. By using iterations (3), (2), condition (1) and using Lemma 2, we have

$$\begin{aligned}
\int_0^{\|u_{n+1}-x_{n+1}\|}\varphi(t)dt &\leq \|u_{n+1}-x_{n+1}\|+a_n \\
&\leq (1-\alpha_n)\left\|u_n-y_n^1\right\|+\alpha_n\left\|Tu_n-Ty_n^1\right\|+a_n \\
&\leq (1-\alpha_n)\int_0^{\left\|u_n-y_n^1\right\|}\varphi(t)dt+\alpha_n\int_0^{\left\|Tu_n-Ty_n^1\right\|}\varphi(t)dt+2a_n \\
&\leq [1-\alpha_n(1-c)]\int_0^{\left\|u_n-y_n^1\right\|}\varphi(t)dt+2a_n
\end{aligned}$$

and using condition (1) and Lemma 2

$$\begin{aligned}
\int_0^{\left\|u_n-y_n^1\right\|}\varphi(t)dt &\leq [1-\beta_n^1(1-c)]\int_0^{\left\|u_n-y_n^2\right\|}\varphi(t)dt+\beta_n^1\int_0^{\|Tu_n-u_n\|}\varphi(t)dt+3a_n \\
\int_0^{\left\|u_n-y_n^2\right\|}\varphi(t)dt &\leq [1-\beta_n^2(1-c)]\int_0^{\left\|u_n-y_n^3\right\|}\varphi(t)dt+\beta_n^2\int_0^{\|Tu_n-u_n\|}\varphi(t)dt+3a_n \\
\int_0^{\left\|u_n-y_n^3\right\|}\varphi(t)dt &\leq [1-\beta_n^3(1-c)]\int_0^{\left\|u_n-y_n^4\right\|}\varphi(t)dt+\beta_n^3\int_0^{\|Tu_n-u_n\|}\varphi(t)dt+3a_n
\end{aligned}$$

and continuing the above process

$$\int_0^{\left\|u_n-y_n^{k-1}\right\|}\varphi(t)dt \leq [1-\beta_n^{k-1}(1-c)]\int_0^{\|u_n-x_n\|}\varphi(t)dt+\beta_n^{k-1}\int_0^{\|Tu_n-u_n\|}\varphi(t)dt+3a_n$$

and we obtain

$$\begin{aligned}\int_0^{\|u_{n+1}-x_{n+1}\|}\varphi(t)dt &\leq [1-\alpha_n(1-c)]\int_0^{\left\|u_n-y_n^1\right\|}\varphi(t)dt+2a_n\\ &\leq [1-\alpha_n(1-c)][1-\beta_n^1(1-c)]\int_0^{\left\|u_n-y_n^2\right\|}\varphi(t)dt+\\ &\quad+\beta_n^1\int_0^{\|Tu_n-u_n\|}\varphi(t)dt+5a_n\\ &\leq [1-\alpha_n(1-c)][1-\beta_n^1(1-c)][1-\beta_n^2(1-c)]\int_0^{\left\|u_n-y_n^3\right\|}\varphi(t)dt+\\ &\quad+\beta_n^1\int_0^{\|Tu_n-u_n\|}\varphi(t)dt+\beta_n^2\int_0^{\|Tu_n-u_n\|}\varphi(t)d+8a_n\\ &\leq \ldots\\ &\leq [1-\alpha_n(1-c)][1-\beta_n^1(1-c)][1-\beta_n^2(1-c)]\ldots[1-\beta_n^{k-1}(1-c)]\cdot\\ &\quad\cdot\int_0^{\|u_n-x_n\|}\varphi(t)dt+\beta_n^1\int_0^{\|Tu_n-u_n\|}\varphi(t)dt+\beta_n^2\int_0^{\|Tu_n-u_n\|}\varphi(t)d+\ldots+\\ &\quad+\beta_n^{k-1}\int_0^{\|Tu_n-u_n\|}\varphi(t)d+(3k+2)a_n\end{aligned}$$

and since $\lim\limits_{n\to\infty}\int_0^{\|u_n-p\|}\varphi(t)dt=0$ and

$$\begin{aligned}\int_0^{\|Tu_n-u_n\|}\varphi(t)dt &\leq \|Tu_n-u_n\|+k_n\\ &\leq \int_0^{\|Ty_n-p\|}\varphi(t)dt+\int_0^{\|u_n-p\|}\varphi(t)dt+2a_n\\ &\leq (1+c)\int_0^{\|u_n-p\|}\varphi(t)dt+2a_n\end{aligned}$$

Then $\lim\limits_{n\to\infty}\int_0^{\|Tu_n-u_n\|}\varphi(t)dt=0$.

Now we will prove $(ii)\Rightarrow(i)$.Using condition (1) and Lemma 2, we have

$$\begin{aligned}\int_0^{\|x_{n+1}-u_{n+1}\|}\varphi(t)dt &\leq \|x_{n+1}-u_{n+1}\|+a_n\\ &\leq (1-\alpha_n)\left\|y_n^1-u_n\right\|+\alpha_n\left\|Ty_n^1-Tu_n\right\|+a_n\\ &\leq [1-\alpha_n(1-c)]\int_0^{\left\|y_n^1-u_n\right\|}\varphi(t)dt+2a_n\end{aligned}$$

and

$$\begin{aligned}
\int_0^{\|y_n^1-u_n\|}\varphi(t)dt &\leq \int_0^{\|y_n^2-u_n\|}\varphi(t)dt+\beta_n^1\int_0^{\|Ty_n^2-y_n^2\|}\varphi(t)dt+3a_n\\
\int_0^{\|y_n^2-u_n\|}\varphi(t)dt &\leq \int_0^{\|y_n^3-u_n\|}\varphi(t)dt+\beta_n^2\int_0^{\|Ty_n^3-y_n^3\|}\varphi(t)dt+3a_n\\
\int_0^{\|y_n^3-u_n\|}\varphi(t)dt &\leq \int_0^{\|y_n^4-u_n\|}\varphi(t)dt+\beta_n^3\int_0^{\|Ty_n^4-y_n^4\|}\varphi(t)dt+3a_n\\
&\cdots\\
\int_0^{\|y_n^{k-1}-u_n\|}\varphi(t)dt &\leq \int_0^{\|x_n-u_n\|}\varphi(t)dt+\beta_n^{k-1}\int_0^{\|Tx_n-x_n\|}\varphi(t)dt+3a_n
\end{aligned}$$

From Teorem 2 it follows that $\lim_{n\to\infty}\int_0^{\|x_n-p\|}\varphi(t)dt=0$. Since T satisfies condition (1) and T has fixed point p.

$$\begin{aligned}
\int_0^{\|Tx_n-x_n\|}\varphi(t)dt &\leq \|x_n-p\|+\|Tx_n-p\|+a_n\\
&\leq \int_0^{\|x_n-p\|}\varphi(t)dt+\int_0^{\|Tx_n-p\|}\varphi(t)dt+3a_n\\
&\leq (1+c)\int_0^{\|x_n-p\|}\varphi(t)dt+3a_n
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{\|Ty_n^2-y_n^2\|}\varphi(t)dt &\leq \|y_n^2-p\|+\|Ty_n^2-p\|+a_n\\
&\leq (1+c)\int_0^{\|y_n^2-p\|}\varphi(t)dt+3a_n\\
&\leq (1+c)[1-\beta_n^2(1-c)]\int_0^{\|y_n^3-p\|}\varphi(t)dt+5a_n\\
&\leq (1+c)[1-\beta_n^2(1-c)][1-\beta_n^3(1-c)]\int_0^{\|y_n^4-p\|}\varphi(t)dt+7a_n\\
&\leq \cdots\\
&\leq (1+c)[1-\beta_n^2(1-c)]...[1-\beta_n^{k-1}(1-c)]\int_0^{\|x_n-p\|}\varphi(t)dt+(2k+1)a_n \qquad (19)
\end{aligned}$$

where $k=1,2,...,k$ is a finete and from (19), Lemma 1 and Theorem 1. We have

$$\begin{aligned}
\lim_{n\to\infty}a_n &= \lim_{n\to\infty}\int_0^{\|x_n-p\|}\varphi(t)dt=\lim_{n\to\infty}\int_0^{\|Tx_n-x_n\|}\varphi(t)dt=\lim_{n\to\infty}\int_0^{\|Ty_n^2-y_n^2\|}\varphi(t)dt=\\
&= ...=\lim_{n\to\infty}\int_0^{\|Ty_n^{k-1}-y_n^{k-1}\|}\varphi(t)dt=0.
\end{aligned}$$

The proof is complete. ■

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