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Examples of Gradient Ricci Solitons on 4-Dimensional Riemannian Manifold

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Abstract. In this present paper, we provide two examples of gradient Ricci solitons on 4-dimensional Riemannian manifold showing for the existence.

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1. Introduction

In 1982, Hamilton [7] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman ([11], [12]) used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold (M,g) defined as follows:

$$\frac{\partial}{\partial t}g(t) = -2Ric,$$

where Ric is the Ricci tensor.
A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton (g,V,ρ) on a Riemannian manifold (M,g) is a generalization of an Einstein metric such that [8]

$$\frac{1}{2}\pounds_V g + Ric = \rho g, \tag{1}$$

where $\pounds_V$ is the Lie derivative operator along the vector field V on M and ρ is a real number. The Ricci soliton is said to be shrinking, steady and expanding according as ρ is negative, zero and positive respectively. Moreover, if the vector field V is the gradient of some smooth function f (called potential function) on M then the Riemannian manifold (M,g) is said to be gradient Ricci solition. A Riemannian manifold (M,g) is called a gradient Ricci soliton [1] if there exists a smooth function $f : M \to \mathbb{R}$, sometimes called potential function, satisfying:

$$R_{ij} + f_{,ij} = \rho g_{ij} \tag{2}$$

where ρ is a real number and R_{ij} are the components of the Ricci tensor. The gradient Ricci solitons have been studied by several authors such as [3], [6], [10] and many others. In this connection it may be mentioned that many authors studied various spaces and spacetime such as [2], [4], [5], [9], [13], [14], [17]. In the present paper we provide two examples of gradient Ricci solitons on 4-dimensional Riemannian manifold showing for the existence. The paper is organized as follows. Section 2 is concerned as follows. Section 3 provides the examples of gradient Ricci solitons on 4-dimensional Riemannian manifold.

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2. Preliminaries

This section deals with some preliminaries, which will be required in the sequel.
Let (M,g) be an n-dimensional Riemannian manifold and $(U,x^1,x^2,\cdots,x^n)$ be a coordinate chart on M. The Christoffel symbols of the Levi-Civita connection is denoted by Γ^k_{ij}, is defined by [1]

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right).$$

Using the Christoffel symbols, the components of the Riemann curvature tensor R can be expressed in the following form [1]:

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ki}}{\partial x^j} - \frac{\partial \Gamma^l_{ji}}{\partial x^k} + \Gamma^r_{ki}\Gamma^l_{jr} - \Gamma^r_{ji}\Gamma^l_{kr},$$

while the Ricci tensor (Ric) is defined by $R_{ij} = R^l_{ilk}$.
If $f : M \to \mathbb{R}$ is a smooth function, then we consider [1]:

$$f_{,i} = \frac{\partial f}{\partial x^i}, \quad f_{,ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^m_{ij} f_{,m}. \tag{3}$$

3. Examples of gradient Ricci solitons on 4D Riemannian manifold

In this section we construct the examples of gradient Ricci solitons on 4D Riemannian manifold.
Example 3.1. Let us consider $M = \mathbb{R}^4$ be a manifold endowed with the metric g defined by

$$ds^2 = g_{ij}dx^i dx^j = e^{2x^1}(dx^1)^2 + \sin^2 x^1[(dx^2)^2 + (dx^3)^2 + (dx^4)^2],\ (i,j = 1,2,3,4), \tag{4}$$

where $0 < x^1 < \frac{\pi}{2}$ but $x^1 \neq \frac{\pi}{4}$ [16]. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor R_{ij}, $(i,j = 1,2,3,4)$ and scalar curvature R are

$$\Gamma^1_{11} = 1, \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \cot x^1, \Gamma^1_{22} = -\frac{\sin 2x^1}{2e^{2x^1}} = \Gamma^1_{33} = \Gamma^1_{44}, \tag{5}$$

$$\begin{gathered} R_{1221} = -\sin^2 x^1(1+\cot x^1) = R_{1331} = R_{1441}, R_{2332} = \frac{\sin^2 x^1 \cos^2 x^1}{e^{2x^1}} = R_{2442} = R_{3443}, \\ R_{22} = \frac{2\cos^2 x^1 - \sin^2 x^1(1+\cot x^1)}{e^{2x^1}} = R_{33} = R_{44}, R_{11} = -3(1+\cot x^1), \\ R = \frac{6(\cot^2 x^1 - \cot x^1 - 1)}{e^{2x^1}} \neq 0, \end{gathered} \tag{6}$$

provided $(\cot^2 x^1 - \cot x^1 - 1) \neq 0$ and the components which can be obtained from these by the symmetry properties. Therefore $\mathbb{R}^4$ with the considered metric is a Riemannian manifold (M^4, g) of non-vanishing scalar curvature [16].
Next, for an arbitrary function $f(x^1,x^2,x^3,x^4)$ on M, we compute

$$\begin{cases} f_{,11} = \frac{\partial^2 f}{\partial (x^1)^2} - \frac{\partial f}{\partial x^1}; f_{,12} = f_{,21} = \frac{\partial^2 f}{\partial x^1 \partial x^2} - \cot x^1 \frac{\partial f}{\partial x^2}; \\ f_{,13} = f_{,31} = \frac{\partial^2 f}{\partial x^1 \partial x^3} - \cot x^1 \frac{\partial f}{\partial x^3}; f_{,14} = f_{,41} = \frac{\partial^2 f}{\partial x^1 \partial x^4} - \cot x^1 \frac{\partial f}{\partial x^4}; \\ f_{,22} = \frac{\partial^2 f}{\partial (x^2)^2} + \frac{\sin 2x^1}{2e^{2x^1}} \frac{\partial f}{\partial x^1}; f_{,23} = f_{,32} = \frac{\partial^2 f}{\partial x^2 \partial x^3}; f_{,24} = f_{,42} = \frac{\partial^2 f}{\partial x^2 \partial x^4}; \\ f_{,33} = \frac{\partial^2 f}{\partial (x^3)^2} + \frac{\sin 2x^1}{2e^{2x^1}} \frac{\partial f}{\partial x^1}; f_{,34} = f_{,43} = \frac{\partial^2 f}{\partial x^3 \partial x^4}; \\ f_{,44} = \frac{\partial^2 f}{\partial (x^4)^2} + \frac{\sin 2x^1}{2e^{2x^1}} \frac{\partial f}{\partial x^1}. \end{cases} \tag{7}$$

With (6) and (7), the relation (2) reduces to the following equations:

$$R_{11}+f_{,11}=\rho g_{11}, \tag{8}$$
$$R_{12}+f_{,12}=\rho g_{12}, \tag{9}$$
$$R_{13}+f_{,13}=\rho g_{13}, \tag{10}$$
$$R_{14}+f_{,14}=\rho g_{14}, \tag{11}$$
$$R_{22}+f_{,22}=\rho g_{22}, \tag{12}$$
$$R_{23}+f_{,23}=\rho g_{23}, \tag{13}$$
$$R_{24}+f_{,24}=\rho g_{24}, \tag{14}$$
$$R_{33}+f_{,33}=\rho g_{33}, \tag{15}$$
$$R_{34}+f_{,34}=\rho g_{34}, \tag{16}$$
$$R_{44}+f_{,44}=\rho g_{44}. \tag{17}$$

Using (6) and (7), we have from the relations (9), (10), (11), (13), (14) and (16) that $\frac{\partial f}{\partial x^2}=\frac{\partial f}{\partial x^3}=\frac{\partial f}{\partial x^4}=0$, i.e. f is a function of x^1 only.
Also in view of (6) and (7) we obtain from (12), (15) and (17) that

$$\frac{\partial f}{\partial x^1}=\frac{2\rho\sin^2 x^1 e^{2x^1}-4\cos^2 x^1+2\sin^2 x^1(1+\cot x^1)}{\sin 2x^1} \tag{18}$$

and hence

$$\begin{aligned}\frac{\partial^2 f}{\partial (x^1)^2} &= \frac{1}{\sin^2 2x^1}\Big[\big\{2\rho(\sin 2x^1+2\sin^2 x^1)e^{2x^1}+4\sin 2x^1\\ &+ 2\sin 2x^1(1+\cot x^1)-2\big\}\sin 2x^1\\ &- \big\{2\rho\sin^2 x^1 e^{2x^1}-4\cos^2 x^1+2\sin^2 x^1(1+\cot x^1)\big\}2\cos 2x^1\Big].\end{aligned} \tag{19}$$

By virtue of (6), (7) and (18) it follows from (8) that

$$\begin{aligned}\frac{\partial^2 f}{\partial (x^1)^2} &= [\rho e^{2x^1}+3(1+\cot x^1)]\\ &+ \frac{[2\rho\sin^2 x^1 e^{2x^1}-4\cos^2 x^1+2\sin^2 x^1(1+\cot x^1)]}{\sin 2x^1}.\end{aligned} \tag{20}$$

From (19) and (20) by identification we get

$$\rho\sin^3 x^1(\cos x^1-\sin x^1)=0,$$

which implies that $\rho=0$, since $x^1\neq 0,\frac{\pi}{4}$. Consequently (18) yields

$$\begin{aligned}\frac{\partial f}{\partial x^1} &= \frac{2\sin^2 x^1(1+\cot x^1)-4\cos^2 x^1}{\sin 2x^1}\\ &= \tan x^1+1-2\cot x^1\end{aligned} \tag{21}$$

and hence

$$f=\log(\sec x^1)+x^1-2\log(\sin x^1)+a;\;\; a\in\mathbb{R}. \tag{22}$$

This leads to the following:

Theorem 1. Let (M,g) be a Riemannian manifold endowed with the metric given in *(4)*. Then (M,g) is a gradient Ricci soliton having $f(x^1,x^2,x^3,x^4)=\log(\sec x^1)+x^1-2\log(\sin x^1)+a$; $a\in\mathbb{R}$ as potential function.

Example 3.2. Let $M = \mathbb{R}^4$ be a manifold endowed with the metric g defined by

$$ds^2 = g_{ij}dx^i dx^j = e^{x^2}\left[(dx^1)^2 + e^{x^1}(dx^2)^2\right] + (dx^3)^2 + (dx^4)^2,\ (i,j = 1,2,3,4). \tag{23}$$

Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor R_{ij}, $(i,j = 1,2,3,4)$ and scalar curvature R are

$$\Gamma^2_{11} = -\frac{1}{2e^{x^1}}, \Gamma^1_{22} = -\frac{1}{2}e^{x^1}, \Gamma^2_{22} = \frac{1}{2} = \Gamma^1_{12} = \Gamma^2_{12}, \tag{24}$$

$$\begin{gathered} R_{1221} = \frac{1}{4}e^{x^1+x^2}, \\ R_{11} = \frac{1}{4}, R_{22} = \frac{1}{4}e^{x^1}, \\ R = \frac{1}{2e^{x^2}} \neq 0, \end{gathered} \tag{25}$$

and the components which can be obtained from these by the symmetry properties. Therefore $\mathbb{R}^4$ with the considered metric is a Riemannian manifold (M^4, g) of non-vanishing scalar curvature [15].
Next, for an arbitrary function $f(x^1,x^2,x^3,x^4)$ on M, we compute

$$\begin{cases} f_{,11} = \frac{\partial^2 f}{\partial(x^1)^2} + \frac{1}{2e^{x^1}}\frac{\partial f}{\partial x^2}; f_{,12} = f_{,21} = \frac{\partial^2 f}{\partial x^1 \partial x^2} - \frac{1}{2}\frac{\partial f}{\partial x^1} - \frac{1}{2}\frac{\partial f}{\partial x^2}; \\ f_{,13} = f_{,31} = \frac{\partial^2 f}{\partial x^1 \partial x^3}; f_{,14} = f_{,41} = \frac{\partial^2 f}{\partial x^1 \partial x^4}; \\ f_{,22} = \frac{\partial^2 f}{\partial(x^2)^2} + \frac{1}{2}e^{x^1}\frac{\partial f}{\partial x^1} - \frac{1}{2}\frac{\partial f}{\partial x^2}; \\ f_{,23} = f_{,32} = \frac{\partial^2 f}{\partial x^2 \partial x^3}; f_{,24} = f_{,42} = \frac{\partial^2 f}{\partial x^2 \partial x^4}; f_{,33} = \frac{\partial^2 f}{\partial(x^3)^2}; \\ f_{,34} = f_{,43} = \frac{\partial^2 f}{\partial x^3 \partial x^4}; f_{,44} = \frac{\partial^2 f}{\partial(x^4)^2}. \end{cases} \tag{26}$$

With (24) and (25), the relation (2) reduces to the equations (8)–(17). By virtue of (24)–(26) it follows from the relations (8)–(17) that $\rho = 0$ and f must be a constant function.
This leads to the following:

Theorem 2. Let (M,g) be a Riemannian manifold endowed with the metric given in *(23)*. Then (M,g) is a gradient Ricci soliton having constant potential function.

References

[1] Bercu, G., Postolache, M., "Classes of gradient Ricci solitons", *Int. J. Geom. Methods Mod. Phys.,* 8(4), 783–796, 2011.

[2] Büyükkütük, S., Kisi, İ., Mishra, V. N., Öztürk, G., "Some characterizations of curves in Galilean 3-space $\mathbb{G}_3$", *Facta Univ. (Nis) Ser. Math. Inform.,* 31(2), 503–512, 2016.

[3] Cao, H. D., "Geometry of Ricci solitons", *Chinese Ann. Math. Ser. B*, 27, 121–142, 2006.

[4] Deepmala, Mishra, L. N., "Differential operators over modules and rings as a path to the generalized differential geometry", *Facta Univ. (Nis) Ser. Math. Inform.,* 30(5), 753–764, 2015.

[5] Drachal, K., Vandana, "Some algebraic aspects of the gluing of differential spaces", *Georgian Math. J.*, to be appeared.

[6] Eminenti, M., La N., Mantegazza, C., "Ricci solitons: the equation point of view", *Manuscripta Math.,* 127(3), 345–367, 2008.

[7] Hamilton, R. S., "Three-manifolds with positive Ricci curvature", *J. Diff. Geom.,* 17, 255–306, 1982.

[8] Hamilton, R. S., "The Ricci flow on surfaces", *Contemp. Math.,* 71, 237–262, 1988.

[9] Kisi, İ., Büyükkütük, S., Deepmala, Öztürk, G., "A $W(k)$-type curves according to parallel transport frame in Euclidean space $\mathbb{E}^4$", *Facta Univ. (Nis) Ser. Math. Inform.,* 31(4), 2016.

[10] Lopez M. F., Garcia-Rio, E., "A remark on compact Ricci solitons", *Math. Ann.,* 340, 893–896, 2008.

[11] Perelman, G., "The entropy formula for the Ricci flow and its geometric applications", *http://arXiv.org/abs/math/0211159*, 1–39, 2002.

[12] Perelman, G., "Ricci flow with surgery on three manifolds", *http://arXiv.org/abs/math/0303109*, 1–22, 2003.

[13] Pişcoran L.-I., Mishra, V. N., "Projectively flatness of a new class of (α,β)-metrics", *Georgian Math. J.,* 14, 2016.
[14] Pişcoran, L.-I., Mishra, V. N., "S-curvature for a new class of (α,β)-metrics", *Revista de la Real Academia de Ciencias Exactas, Fsicas y Naturales. Serie A. Matemticas*, DOI: 10.1007/s13398-016-0358-3, 2016.
[15] Shaikh, A. A., Hui, S. K., "On decomposable weakly conharmonically symmetric manifolds", *Lobachevski J. Math.,* 29, 206–215, 2008.
[16] Shaikh, A. A., Hui, S. K., "On some classes of generalized quasi-Einstein manifolds", *Commun. Korean Math. Soc.,* 24(3), 415–424, 2009.
[17] Vandana, Deepmala, Drachal, K., Mishra, V. N., "Some algebro-geometric aspects of spacetime c-boundary", *Math. Aeterna,* 6, 561–572, 2016.