Modelling & Application & Theory
Vol. 1, No. 1 (2016) 2548–0596

The Numerical Solutions of a Class of Fredholm Integral Equations of Second Kind

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Abstract. In this paper we consider a class of integral equations associated with the invariant mean value property for $\mathscr{M}$-harmonic functions and a class of integral equations associated with the invariant mean value property for hyperbolic-harmonic functions. We use projection methods, Nystrom method to obtain numerical solutions of these integral equations. We present algorithms to obtain numerical solutions of these two classes of Fredholm integral equations of second kind.

Keywords. Fredholm integral equations, Projection method, $\mathscr{M}$-harmonic functions, hyperbolic harmonic functions, Invariant mean value property.

2000 MSC. 65R20, 45B05, 31B05.

1. Introduction and Preliminaries

Ahern, Flores and Rudin [1], Axler, Cuckovic [3] and Yi [10] studied the class of integral equations

$$g(x) = (1-x)^{n+1}\int_0^1 \frac{n+tx}{(1-tx)^{n+2}}g(t)t^{n-1}dt, n\in\mathbb{N} \tag{1}$$

associated with the invariant mean value property of $\mathscr{M}$-harmonic functions. They proved that constants are the only solutions of these integral equations if $n \leq 11$ and this is not true if $n \geq 12$. They pointed out that this class of equations (1) can be studied for all n, not necessarily integers, and then the question arises: At which point between 11 and 12 does the above mentioned integral equation (1) has non-constant solutions? In 1995, Yi [10] gave a characterization of the functions fixed by the integral operator

$$Tg(x) = (1-x)^{n+1}\int_0^1 \frac{n+tx}{(1-tx)^{n+2}}g(t)t^{n-1}dt. \tag{2}$$

In [10], Yi obtained a critical point $11+\varepsilon_0, 0<\varepsilon_0<1$, such that the target equation has only constant solutions if and only if $1 \leq n \leq 11+\varepsilon_0$.

In 2003, Jevtic [8] studied the equation

$$g(x) = (1-x)^{\gamma}\frac{\gamma}{2}\int_0^1 \frac{1+tx}{(1-tx)^{\gamma+1}}g(t)t^{\frac{\gamma}{2}-1}dt, \tag{3}$$

$\gamma \geq 2$, that arises naturally in the study of the invariant mean value property of hyperbolically-harmonic functions and showed that if $\gamma \geq 2$, then the constants are the only solutions of the equation (3) if and only if $2 \leq \gamma \leq 12+\varepsilon_1, 0<\varepsilon_1<1$. The approach to the problem considered in [8] comes from Yi's work [10].

This paper deals with numerical solutions of the integral equations (1) and (3). We use Projection methods, Nystrom method to obtain numerical solutions of these integral equations. We propose algorithms for solving the integral equations (1) and (3). The solutions obtained are expressed in terms of hypergeometric

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functions. The layout of this paper is as follows: In section 2 we make some observations on the behavior of the solution of the integral equation (1) and on the fixed points of the corresponding integral operator T given in (2). Section 3 is devoted to establish that the Laguerre polynomials are associated with the kernel $\frac{1}{(1-ts)^{\alpha+1}}, \alpha \in \mathbb{Z}_+$. In section 4, we use Projection methods and Nystrom method to obtain numerical solutions of these integral equations. In section 5 we present an algorithm to obtain numerical solutions of the integral equation given in (1). We observe that except constants all solutions of the equation are functions of unbounded variations. The solutions are in conformation to the theoretical results. In section 6, we present an algorithm to obtain numerical solutions of the integral equation given in (3).

2. On the solutions of the integral equations

In this section we investigate whether constants are the only solutions of the integral equation (1) in $L^1[0,1]$ for all integers n such that $1 \leq n \leq 11$ and study the behavior of the nonconstant solutions of the integral equation (1) in $L^1[0,1]$ for all $n \in \mathbb{N}, n \geq 12$. The following observations can be made about the solution of the integral operator equation

$$V(t) = (1-t)^{n+1} \int_0^1 \frac{n+ts}{(1-ts)^{n+2}} V(s) s^{n-1} ds \tag{4}$$

for every $t \in (0,1)$. Casting it in a slightly different form, we want to show that if $n \leq 11$ then the eigenspace corresponding to the eigenvalue 1 of the operator T is one dimensional, where

$$(Tu)(t) = (1-t)^{n+1} \int_0^1 \frac{n+ts}{(1-ts)^{n+2}} u(s) s^{n-1} ds \tag{5}$$

defined on $C([0,1)) \cap L^1([0,1])$. The operator T is an integral operator with positive kernel which maps $L^1([0,1])$ to $C([0,1))$, so the domain may as well be considered as $L^1([0,1])$. But the operator T does not leave L^1 invariant because

$$G(s) = \int_0^1 (1-t)^{n+1} \frac{n+ts}{(1-ts)^{n+2}} dt$$

is an unbounded function of s and hence there exists $0 \leq f \in L^1$ such that

$$\int_0^1 f(s) G(s) ds = \infty.$$

We could regard T as an operator on $C([0,1))$, which it clearly leaves invariant. The difficulty here is that there is no natural norm on $C([0,1))$, although there is a metric which makes $C([0,1))$ into a Frechet space. The upshot of this is that it is difficult to find a convenient and natural invariant Banach space for T. For a while consider the case $n = 1$. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $dA(z) = \frac{1}{\pi} dxdy$ be the normalized area measure on $\mathbb{D}$. We know in $\mathbb{D}$, the only measure left invariant by all Mobius transformations is the pseudo-hyperbolic measure $d\eta(z) = \dfrac{dA(z)}{(1-|z|^2)^2}$. Therefore, the only harmonic function in $L^p(\mathbb{D}, d\eta)$ is constant zero. To see this e.g., for $L^2(\mathbb{D}, d\eta)$, denote

$$M(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt. \tag{6}$$

This is a nonnegative and nondecreasing function of r. Further,

$$\|f\|^2_{L^2(\mathbb{D},d\eta)} = \int_0^1 M(r) \frac{2r}{(1-r^2)^2} dr < \infty. \tag{7}$$

So $M(r)$ must tend to zero as $r \to 1$. Thus $M(r) \equiv 0$, whence $f = 0$. Thus even though one can show that each of the space $L^p((0,1), \frac{dt}{(1-t)^2}), 1 \leq p \leq \infty$ is an invariant subspace [1], [5] of T (when $n = 1$) but these spaces are no good in this context (see [1] and [5]). This is because (except for L^∞) the corresponding spaces $L^p(\mathbb{D}, d\eta)$ do not contain nonzero harmonic functions, even no nonzero constants. Similar is the case for $\mathbb{B}_n$, the open unit ball of $\mathbb{C}^n$, $n \in \mathbb{N}$, with respect to the Euclidean metric. The letter ν denote the Lebesgue measure on $\mathbb{C}^n$, normalized so that $\nu(\mathbb{B}_n) = 1$. Consider the space $L^2(\mathbb{B}_n, d\nu)$ for an integer $n \geq 1$. Let $L^2_a(\mathbb{B}_n, d\nu)$ be the corresponding Bergman space and $K_{\mathbb{B}_n}$ is the reproducing kernel for $L^2_a(\mathbb{B}_n, d\nu)$. Note that for $z, \lambda \in \mathbb{B}_n$,

$$K_{\mathbb{B}_n}(z, \lambda) = \frac{n!}{(1 - z \cdot \bar{\lambda})^{n+1}} \tag{8}$$

where $z \cdot \bar{\lambda} = z_1 \bar{\lambda}_1 + \cdots + z_n \bar{\lambda}_n$. Let $d\eta'(z) = K_{\mathbb{B}_n}(z,z) d\nu(z)$. It is not so difficult to check that if $f \in L^2(\mathbb{B}_n, d\eta')$ is $\mathscr{M}$-harmonic then $f \equiv 0$. The argument is as follows: Denote the unit sphere, the boundary of the open unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ by S_n. Let $d\sigma$ be the normalized surface-area measure (Hausdorff measure) of S_n such that $\sigma(S_n) = 1$. Let $M(r) = \int_{S_n} |f(z)|^2 d\sigma(z)$. Then

$$\begin{aligned} \|f\|^2_{L^2(\mathbb{B}_n, d\eta')} &= \int_{\mathbb{B}_n} |f(z)|^2 d\eta'(z) \\ &= \int_0^1 M(r) K_{\mathbb{B}_n}(z,z) 2n r^{2n-1} dr \\ &= n \int_0^1 M(r) n! \frac{t^{n-1}}{(1-t)^{n+1}} dt \qquad \text{where} \quad t = r^2. \end{aligned}$$

So $M(r)$ must tend to zero as $r \to 1$. Thus $M(r) \equiv 0$. Hence since f is $\mathscr{M}$-harmonic, by maximum principle $f \equiv 0$. Thus for $n \geq 1$, even though one can show that the space $L^2((0,1), \frac{t^{n-1} dt}{(1-t)^{n+1}})$ is an invariant subspace [1] of T but these spaces are no good in this context. This is so since the space $L^2(\mathbb{B}_n, d\eta')$ do not contain nonzero harmonic functions, even no nonzero constants. For more details see [1], [3] and [10]. Now we study the behavior of the kernel of the integral operator T defined in (2). First we make the following observation.

Lemma 1. If $r > 0$ and $s > 0$ then the following hold:

1. $\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - rse^{i\theta}|^2} d\theta = \frac{1}{1 - r^2 s^2}$;
2. $\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - rse^{i\theta}|^4} d\theta = \frac{1 + r^2 s^2}{(1 - r^2 s^2)^3}$;
3. $\frac{1}{2\pi} \int_0^{2\pi} \frac{\Re e^{i\theta}}{|1 - rse^{i\theta}|^2} d\theta = \frac{rs}{1 - r^2 s^2}$;
4. $\frac{1}{2\pi} \int_0^{2\pi} \frac{\Re e^{i\theta}}{|1 - rse^{i\theta}|^4} d\theta = \frac{2rs}{(1 - r^2 s^2)^3}$.

Proof: These equalities are quite simple to establish, using the Taylor series for $\frac{1}{1-x}$ and the orthonormality of the exponentials in $L^2[0, 2\pi]$. It is also possible to verify the lemma directly using the residue theorem though the calculations do become tedious. □

One interesting property about the kernel is that the function $\dfrac{n + ts}{(1 - ts)^{n+2}}$ is an increasing function of t for fixed s. This implies that if $u \geq 0$ then $\int_0^1 \frac{n+ts}{(1-ts)^{n+2}} u(s)ds$ is an increasing function of t, so if u is non-negative then $(Tu)(t) = (1-t)^{n+1} f(t)$ where f is increasing. In particular, any nonnegative fixed point of T has this property. One may think that this property might contradict the oscillatory property, but it seems it does not. In the one-dimensional case (when $n = 1$) the function given by

$$f(t) = \frac{1}{(1-t)^2} \{2 + \sin[\log(1-t)]\} \tag{9}$$

is monotone, but $(1-t)^2 f(t)$ oscillates infinitely many times on $[0,1)$. It does, however, attain its supremum and infimium, so it is not quite of the right form. If f is bounded, this does contradict the oscillatory property. May be one could show that the operator does not map any functions at all to functions with this property. Thus we have shown that any fixed point of T (other than constants) oscillates infinitely many times, so all its derivatives also oscillate. Consider now the case $n = 1$. Let $k(s,t) = \frac{1+ts}{(1-ts)^3}$. Then

$$\frac{d^m}{ds^m} \int_0^1 k(s,t)u(t)dt = \int_0^1 [\frac{\partial^m}{\partial s^m} k(s,t)]u(t)dt. \tag{10}$$

Now if $u \geq 0$ and $\frac{\partial^m}{\partial s^m} k(s,t) \geq 0$ then this is nonnegative. Thus, if $\frac{\partial^m}{\partial s^m} k(s,t) \geq 0$ then T has no nonnegative, non-oscillatory fixed points, hence no bounded above or bounded below fixed points at all. Further it may be observed that $\frac{\partial^m}{\partial s^m} k(s,t)$ is not single-signed in a neighborhood of the singularity at least for any m less than about 200. Finally, if k did satisfy the condition that $\frac{\partial^m}{\partial s^m} k(s,t) \geq 0$ for some integer $m \geq 1$ then if u is bounded,

$$\frac{d^m}{ds^m} T(u - \inf u) \geq 0 \text{ which implies } \frac{d^m}{ds^m} Tu \geq 0 \tag{11}$$

and

$$\frac{d^m}{ds^m} T(\sup\ u - u) \geq 0 \text{ which implies } \frac{d^m}{ds^m} Tu \leq 0. \tag{12}$$

So if u is bounded, then Tu is a polynomial of degree $\leq n-1$. That is, the images of bounded functions are equal to polynomials in a neighborhood of 1. Again consider the case $n = 1$. Let

$$T_1 g(x) = (1-x)^2 \int_0^1 \frac{1+tx}{(1-tx)^3} g(t)dt.$$

The operator T_1 is bounded [6] [9] on $L^2[0,1]$ and

$$\begin{aligned}
\langle g, T_1^* f \rangle &= \langle T_1 g, f \rangle \\
&= \int_0^1 T_1 g(x) \overline{f(x)} dx \\
&= \int_0^1 (1-x)^2 \int_0^1 \frac{1+tx}{(1-tx)^3} g(t) \overline{f(x)} dt dx \\
&= \int_0^1 g(t) \left(\int_0^1 (1-x)^2 \frac{1+tx}{(1-tx)^3} \overline{f(x)} dx \right) dt.
\end{aligned}$$

Thus

$$
\begin{aligned}
(T_1^* f)(t) &= \overline{\int_0^1 (1-x)^2 \frac{1+tx}{(1-tx)^3} \overline{f(x)} dx} \\
&= \int_0^1 (1-x)^2 \frac{1+tx}{(1-tx)^3} f(x) dx.
\end{aligned}
$$

Theorem 1. T_1 is not a bounded operator on $L^1[0,1]$.

Proof: The operator T_1 is defined as

$$(T_1 g)(x) = (1-x)^2 \int_0^1 \frac{1+tx}{(1-tx)^3} g(t) dt.$$

Notice that

$$\frac{1+tx}{(1-tx)^3} = \sum_{n=0}^{\infty} (n+1)^2 t^n x^n,$$

$0 < x < 1, 0 < t < 1$. If T_1 were bounded on $L^1[0,1]$, its adjoint $T_1^* = S_1$, where

$$(S_1 f)(t) = \int_0^1 (1-x)^2 \frac{1+tx}{(1-tx)^3} f(x) dx$$

would be a bounded operator on $L^\infty[0,1]$.
Now

$$
\begin{aligned}
(S_1 1)(t) &= \int_0^1 (1-x)^2 \frac{1+tx}{(1-tx)^3} dx \\
&= \int_0^1 (1-x)^2 \sum_{n=0}^{\infty} (n+1)^2 t^n x^n dx \\
&= \sum_{n=0}^{\infty} (n+1)^2 t^n \int_0^1 (1-x)^2 x^n dx \\
&= \sum_{n=0}^{\infty} \frac{2(n+1)}{(n+2)(n+3)} t^n.
\end{aligned}
$$

As $t \to 1$, this expression behaves asymptotically like $log \frac{1}{1-|z|^2}$, hence $S_1 1 \notin L^\infty[0,1]$, so $S_1 = T_1^*$ cannot be a bounded operator on $L^\infty[0,1]$. □

Remark 2.1. Notice that the fixed point $V(t)$ of the operator T_1 satisfies the integral equation

$$V(t) = (1-t)^2 \int_0^1 \frac{1+ts}{(1-ts)^3} V(s) ds. \tag{13}$$

If $m \in \mathbb{Z}_+$,

$$V\left(\frac{1}{m}\right) = (m-1)^2 \int_0^1 \frac{m+s}{(m-s)^3} V(s) ds.$$

Thus $V(1) = 0, V(0) = \int_0^1 V(s) ds$ and $V(\frac{1}{2}) = \int_0^1 \frac{2+s}{(2-s)^3} V(s) ds.$

If constants are the only solutions of the integral equation (13) then $V'(t) = 0$ for all t and for all $V \in C[0,1) \cap L^1[0,1]$ satisfying the integral equation

$$V(t) = (1-t)^2 \int_0^1 \frac{1+ts}{(1-ts)^3} V(s)ds.$$

Notice that

$$\frac{1+ts}{(1-ts)^3} = \frac{2}{(1-ts)^3} - \frac{1}{(1-ts)^2}.$$

Hence

$$\begin{aligned} \frac{d}{dt}\left[\frac{1+ts}{(1-ts)^3}\right] &= 2\left[\frac{-3(1-ts)^2(-s)}{(1-ts)^6}\right] - \frac{(-2)(1-ts)(-s)}{(1-ts)^4} \\ &= \frac{6s}{(1-ts)^4} - \frac{2s}{(1-ts)^3}. \end{aligned}$$

Thus

$$\begin{aligned} V'(t) &= -2(1-t)\int_0^1 \frac{1+ts}{(1-ts)^3} V(s)ds + (1-t)^2 \int_0^1 \left[\frac{6s}{(1-ts)^4} - \frac{2s}{(1-ts)^3}\right] V(s)ds \\ &= 2(1-t)\int_0^1 \left[-\left(\frac{1+ts}{(1-ts)^3}\right) + (1-t)s\left(\frac{3}{(1-ts)^4} - \frac{1}{(1-ts)^3}\right)\right] V(s)ds \\ &= 2(1-t)\int_0^1 \left[-\left(\frac{1+ts}{(1-ts)^3}\right) + (1-t)s\left(\frac{3-1+ts}{(1-ts)^4}\right)\right] V(s)ds \\ &= 2(1-t)\int_0^1 \left[-\left(\frac{1+ts}{(1-ts)^3}\right) + (1-t)s\left(\frac{2+ts}{(1-ts)^4}\right)\right] V(s)ds \\ &= 2(1-t)\int_0^1 \left[\frac{-1+t^2s^2+2s+ts^2-2ts-t^2s^2}{(1-ts)^4}\right] V(s)ds \\ &= 2(1-t)\int_0^1 \left(\frac{-1+2s-2ts+ts^2}{(1-ts)^4}\right) V(s)ds \\ &= 2(1-t)\int_0^1 \frac{2(1-ts)-s(1-ts)+3(s-1)}{(1-ts)^4} V(s)ds \\ &= 2(1-t)\int_0^1 \frac{2}{(1-ts)^3} V(s)ds - 2(1-t)\int_0^1 \frac{s}{(1-ts)^3} V(s)ds \\ &+ 2(1-t)\int_0^1 \frac{3s-3}{(1-ts)^4} V(s)ds. \end{aligned}$$

Therefore $V'(t) = 0$ if and only if

$$\int_0^1 \frac{2-s}{(1-ts)^3} V(s)ds = \int_0^1 \frac{3(1-s)}{(1-ts)^4} V(s)ds.$$

□

If $0 < t < 1, 0 < s < 1$, then it is easy to check that

$$\frac{1+ts}{(1-ts)^3} = \sum_{m=0}^{\infty} (m+1)^2 t^m s^m.$$

Thus

$$\begin{aligned}(1-x)^2\int_0^1 \frac{1+tx}{(1-tx)^3}dt &= (1-x)^2\int_0^1\left(\sum_{m=0}^{\infty}(m+1)^2t^mx^m\right)dt\\ &= (1-x)^2\sum_{m=0}^{\infty}\left[(m+1)^2x^m\frac{t^{m+1}}{m+1}\right]_0^1\\ &= (1-x)^2\sum_{m=0}^{\infty}(m+1)x^m.\end{aligned}$$

3. Laguerre differential equation

In this section we relate the Laguerre polynomials with the kernel $\frac{1}{(1-ts)^{\alpha+1}}, \alpha\in\mathbb{Z}_+$. The differential equation

$$x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+\lambda y=0 \tag{14}$$

where λ is a real constant is called the Laguerre's differential equation. It can be solved by Frobenius method. Let us assume that $y=\sum_{r=0}^{\infty}a_rx^{k+r}$ is the solution of the given equation (14). Substituting the series for y, y' and y'' in (14) and equating the coefficients of various powers of x to 0 we obtain

$$y=a_0\left[1-\lambda x+\frac{\lambda(\lambda-1)}{(2!)^2}x^2+\cdots+(-1)^r\frac{\lambda(\lambda-1)\cdots(\lambda-r+1)}{(r!)^2}x^r+\cdots\right]. \tag{15}$$

If $\lambda=n\in\mathbb{Z}_+$, then $y=a_0\sum_{r=0}^{n}(-1)^r\frac{n(n-1)\cdots(n-r+1)}{(r!)^2}x^r$. We define the standard solution of the Laguerre equation $x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+ny=0$ as that for which $a_0=1$ and call it the Laguerre polynomial of order n, denote it by $L_n(x)$. Thus $L_n(x)=\sum_{r=0}^{n}(-1)^r\frac{n!}{(n-r)!(r!)^2}x^r=\frac{e^x}{n!}\frac{d^n}{dx^n}(x^ne^{-x})$.

Lemma 2. If $L_n(x)$ is the Laguerre polynomial of order n, then

$$\sum_{m,n=0}^{\infty}e^{-x}t^ns^mL_n(x)L_m(x)=e^{-x}\frac{1}{(1-t)(1-s)}e^{-\frac{xt}{1-t}}e^{-\frac{xs}{1-s}} \tag{16}$$

and

$$\int_0^{\infty}e^{-x}\frac{1}{(1-t)(1-s)}e^{-\frac{xt}{1-t}}e^{-\frac{xs}{1-s}}dx=\frac{1}{1-ts}. \tag{17}$$

Proof: It is not difficult to verify that $\sum_{n=0}^{\infty}t^nL_n(x)=\frac{1}{1-t}e^{-\frac{xt}{1-t}}$. Thus

$$\sum_{m,n=0}^{\infty}e^{-x}t^ns^mL_n(x)L_m(x)=e^{-x}\frac{1}{(1-t)(1-s)}e^{-\frac{xt}{1-t}}e^{-\frac{xs}{1-s}}$$

and

$$\begin{aligned}\int_0^\infty e^{-x}\frac{1}{(1-t)(1-s)}e^{-\frac{xt}{1-t}}e^{-\frac{xs}{1-s}}dx &= \frac{1}{(1-t)(1-s)}\int_0^\infty e^{-x[1+\frac{t}{1-t}+\frac{s}{1-s}]}dx\\ &= \frac{1}{(1-t)(1-s)}\frac{(-1)}{1+\frac{t}{1-t}+\frac{s}{1-s}}e^{-x[1+\frac{t}{1-t}+\frac{s}{1-s}]}\Big|_0^\infty\\ &= \frac{(-1)}{(1-t)(1-s)}\\ &* \frac{(1-t)(1-s)}{(1-t)(1-s)+t(1-s)+s(1-t)}[0-1]\\ &= \frac{1}{1-ts}.\Box\end{aligned}$$

The differential equation $x\dfrac{d^2y}{dx^2}+(\alpha+1-x)\dfrac{dy}{dx}+ny=0, \alpha\in\mathbb{Z}_+$ is called associate Laguerre equation. It is easy to see that if γ is a solution of the Laguerre equation of order $n+\alpha$ given by

$$x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+(n+\alpha)y=0 \tag{18}$$

then $\dfrac{d^\alpha\gamma}{dx^\alpha}$ satisfies Laguerre's associated equation. Thus we define the associated Laguerre polynomial by

$$L_n^\alpha(x)=(-1)^\alpha\frac{d^\alpha}{dx^\alpha}L_{n+\alpha}(x)=\sum_{r=0}^{n}(-1)^r\frac{(n+\alpha)!}{(n-r)!(\alpha+r)!r!}x^r. \tag{19}$$

Lemma 3. If $L_n^\alpha(x)$ is the associated Laguerre polynomial then

$$\sum_{m,n=0}^{\infty}e^{-x}t^ns^mx^\alpha L_n^\alpha(x)L_m^\alpha(x)=\frac{x^\alpha}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}}e^{-\frac{x(1-ts)}{(1-t)(1-s)}} \tag{20}$$

and

$$\int_0^\infty\frac{x^\alpha}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}}e^{-\frac{x(1-ts)}{(1-t)(1-s)}}dx=\frac{\alpha!}{(1-ts)^{\alpha+1}}. \tag{21}$$

Proof: It is not difficult to verify that $\sum_{n=0}^{\infty}t^nL_n^\alpha(x)=\dfrac{1}{(1-t)^{\alpha+1}}e^{-\frac{xt}{1-t}}$. Thus

$$\begin{aligned}\sum_{m,n=0}^{\infty}e^{-x}t^ns^mx^\alpha L_n^\alpha(x)L_m^\alpha(x) &= \frac{e^{-\alpha}x^\alpha}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}}e^{-\frac{xt}{1-t}}e^{-\frac{xs}{1-s}}\\ &= \frac{x^\alpha}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}}e^{-x[1+\frac{t}{1-t}+\frac{s}{1-s}]}\\ &= \frac{x^\alpha}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}}e^{-\frac{x(1-ts)}{(1-t)(1-s)}}\end{aligned}$$

and

$$\begin{aligned}I &= \int_0^\infty\frac{x^\alpha}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}}e^{-\frac{x(1-ts)}{(1-t)(1-s)}}dx\\ &= \frac{1}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}}\int_0^\infty x^\alpha e^{-\frac{x(1-ts)}{(1-t)(1-s)}}dx.\end{aligned}$$

Integrating by parts taking x^α as the first function we obtain

$$\begin{aligned} I &= \frac{1}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}}\left[\left\{-x^\alpha \frac{(1-t)(1-s)}{1-ts} e^{-\frac{(1-ts)x}{(1-t)(1-s)}}\right\}\Big|_0^\infty \right. \\ &+ \left. \alpha \frac{(1-t)(1-s)}{1-ts}\int_0^\infty x^{\alpha-1} e^{-\frac{(1-ts)x}{(1-t)(1-s)}}dx\right] \\ &= \frac{1}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}} \frac{\alpha(1-t)(1-s)}{(1-ts)}\int_0^\infty x^{\alpha-1} e^{-\frac{(1-ts)x}{(1-t)(1-s)}}dx. \end{aligned}$$

Proceeding similarly $(\alpha-1)$ times, we have

$$\begin{aligned} I &= \frac{1}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}} \frac{\alpha(\alpha-1)\cdots 2\cdot 1\cdot (1-t)^\alpha (1-s)^\alpha}{(1-ts)^\alpha}\int_0^\infty e^{-\frac{(1-ts)x}{(1-t)(1-s)}}dx \\ &= \frac{\alpha!}{(1-t)(1-s)(1-ts)^\alpha}\left[-\frac{(1-t)(1-s)}{(1-ts)} e^{-\frac{(1-ts)x}{(1-t)(1-s)}}\right]_0^\infty. \\ &= \frac{\alpha!}{(1-ts)^{\alpha+1}}. \end{aligned}$$

The Lemma follows. □

4. Solution of the integral equation using projection method

We are trying to look at the multiplicity of the eigenvalue of the operator T as n increases past 11. In a sense this is impossible; numerical methods can never see such fine structure, because it occurs on sets with no interior, equivalently, arbitrarily small perturbations can change multiplicities of eigenvalues. The best possible one can do is to look at the separation between the eigenvalues; if the numerics return two or more eigenvalues very close together, that might be because of a multiple eigenvalue in the original problem.

The kernel function has a singularity at $t = x = 1$. We tried to get rid of it, by similarity transformations and by subtracting the singularity, but did not really get anywhere. As long as the singularity remains in place, quadrature methods are doomed. On the other hand, the images of the monomials t^m under the operator (the "kernel moments") are represented by a reasonably simple explicit formula in terms of the hypergeometric functions. This suggests that may be some kind of projection/expansion/Galerkin method would be effective.

The integral operator that is under our consideration is the operator T defined in (1). The general idea is to choose some suitable function space X, then take a finite-dimensional space U and let P be a projection of X onto U, so PT maps U to itself and can therefore be represented by a finite matrix. If U is suitably chosen, then T and PT will have similar properties. The good thing about this approach is that the singularity is integrated out of the way.

We shall proceed with the method given in Chatelin [4]. The integral operator $T : L^\infty[0,1] \longrightarrow L^\infty[0,1]$ is defined as

$$(Tf)(x) = \int_0^1 \frac{(1-x)^{n+1}(n+tx)}{(1-tx)^{n+2}} t^{n-1} f(t)dt, n \in \mathbb{Z}, n \geq 1.$$

Let $\{t_0, t_1, \cdots, t_N\}$ be a partition of $[0,1]$ and let U be the subspace consisting of all functions which are constant on $[t_{j-1}, t_j)$ for all j, that is, the set of characteristic step functions. We define $e_j(t) = \chi_{[t_{j-1},t_j)}$, the characteristic function of the interval $[t_{j-1}, t_j)$. Now $\{e_j(t)\}$ forms a basis for U. Define a projection P from

$L^\infty[0,1]$ onto U by $(Pf)(x) = \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} f(x)dx$, the average of f over the interval $[t_{j-1}, t_j)$ containing t. Then

$$(PTf)(x) = \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \int_0^1 \frac{(1-x)^{n+1}(n+tx)}{(1-tx)^{n+2}} t^{n-1} f(t)dt.$$

Thus the matrix of PT is

$$a_{ij} = \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} \frac{(1-x)^{n+1}(n+tx)}{(1-tx)^{n+2}} t^{n-1} dtdx \tag{22}$$

where $i, j \in \{0, 1, 2, \cdots, N\}$. That is, the matrix of PT contains (apart from scale factors) the integrals of the kernel over the rectangles $(t_{j-1}, t_j) \times (t_{i-1}, t_i)$. These can be calculated quite explicitly and the resulting matrix handled by standard methods. As the mesh partition tends to 0, one might hope that properties of PT converge to the corresponding properties of T. For the partition $0, \frac{1}{N}, \frac{2}{N}, \cdots, 1$, the matrix entries will be

$$a_{ij} = N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{j-1}{N}}^{\frac{j}{N}} \frac{(1-x)^{n+1}(n+tx)}{(1-tx)^{n+2}} t^{n-1} dtdx.$$

We put all these steps in the form of an algorithm (Algorithm 1).

Algorithm 1 Projection Method on the space of piecewise-constant functions

Require: $(Tf)(x) = \int_0^1 \frac{(1-x)^{n+1}(n+tx)}{(1-tx)^{n+2}} t^{n-1} f(t)dt, n \in \mathbb{Z}, n \geq 1.$

Ensure: Eigenvalues of T

for i=1 to N **do**

for j=1 to N **do**

$$a_{ij} = N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{j-1}{N}}^{\frac{j}{N}} \frac{(1-x)^{n+1}(n+tx)}{(1-tx)^{n+2}} t^{n-1} dtdx$$

end for

end for

Compute the eigenvalues of the matrix $A = (a_{ij})_{1 \leq i,j \leq N}$.

Implementing the Algorithm 1 in MATLAB for a sample size $N = 30$, we obtain the eigenvalues for $n = 11$ and 12 as given in the Table 1. On the face of it, there is no evidence here of the multiplicity of the eigenvalue

TABLE 1. Eigenvalues of T

n=11	n=12
0.9999375883	0.999789679
0.01674 + 0.03244i	0.0245243297
0.01674 - 0.03244i	-0.0085 + 0.01480i
-0.02488761	0.00855 - 0.01480i
-0.00366036	-0.008599513
-0.0013 + 0.00176i	0.00139 + 0.00301i

1 as n increases from 1 to 11 and changing from 11 to 12. We recreated the programme in MAPLE and got

slightly different answers; then we increased the number of samples to 50 and got different answers again. Here the problem is basically one of poor convergence. We will need a lot more samples to get a good answer, which ultimately might not be practical. Even creating the matrix in a stable way is a bit tricky; at least in MAPLE, the symbolic integrator handles the t integral fine, but then creates a large number of terms in evaluating the x integral. Summing these gives a certain amount of numerical instability. We wonder if it would be more stable to do only the t integral symbolically, then use a good numerical method on the x integral ?

The general idea is to increase the number of samples in each case and follow the convergence of the second largest (in magnitude) eigenvalue to a limit. It should stabilize. If it does not then there is a problem. If 1 is a non-simple eigenvalue (of the integral operator (2)), then one should see the second eigenvalue converging to 1 as the number of sample grows; if it is simple, it should converge to something else.

We next consider another subspace M of $C(0,1)$ which consists of piecewise linear functions. We restrict the integral operator T corresponding to the integral equation (1) to the space M and find the eigenvalues of $T|_M$. We then check the multiplicities of eigenvalue 1 as sample size N increases. Let $X = C(0,1)$. The interval $[0,1]$ is divided in to $N-1$ intervals with the partitioning points $\{t_i, i = 1,2,\cdots,N\}$ with $t_1 = 0, t_N = 1$. Let M be the subspace of X consisting of piecewise linear functions on $[0,1]$, each function being linear on every subinterval $[t_i,t_{i+1}]$, $i = 1,\cdots,N-1$. For $i = 1,\cdots,N$, let e_i be the piecewise linear function taking value 1 at t_i and 0 at all other points $t_j \neq t_i$. The set $\{e_i\}_{i=1}^N$ forms a basis for M. Let π_N is the piecewise linear interpolatory projection at the points $t_i, i = 1,\cdots,N$, defined by

$$\pi_N x = \sum_{i=1}^{N} x(t_i)e_i. \tag{23}$$

It is easy to see that $\pi_N \to 1$ in projection if $\max\limits_{2\leq i\leq N} |t_i - t_{i-1}| \to 0$. Let $K(t,s) = \dfrac{(1-t)^{n+1}(n+ts)}{(1-ts)^{n+2}} s^{n-1}$ and

$$(Tx)(t) = \int_0^1 K(t,s)x(s)ds.$$

The above integral equation becomes $Tx = x, x \neq 0$. The associated matrix is defined by

$$a_{ij} = \int_{t_{j-1}}^{t_{j+1}} K(t_i,s)e_j(s)ds, i,j = 1,\cdots,N \tag{24}$$

with $t_0 = 0, t_{N+1} = 1$.

Algorithm 2 Projection method on the space of piecewise-linear functions

Require: $(Tf)(x) = \displaystyle\int_0^1 \frac{(1-x)^{n+1}(n+tx)}{(1-tx)^{n+2}} t^{n-1} f(t)dt, n \in \mathbb{Z}, n \geq 1.$

Ensure: Eigenvalues of T

for i=1 to N **do**

for j=1 to N **do**

$$a_{ij} = \int_{t_{j-1}}^{t_{j+1}} K(t_i,s)e_j(s)ds$$

end for

end for

Compute eigenvalues of the matrix $A = (a_{ij})_{1\leq i,j\leq N}$.

This is collocation method [4] at the collocation points $t_i, i = 1 \cdots N$, with the basis functions $e_i, i = 1 \cdots N$. We implemented the above algorithm in MATLAB/MAPLE and obtained the following sets of eigenvalues for different values of n (see Table 2).
From this method we cannot infer whether 1 is an eigenvalue of T and about its multiplicity. This method should work for a large number of sample which ultimately might not be practical.

TABLE 2. Eigenvalues of T

n=11	n=12
0.0350	0.0286
0.0060	0.0030
0.0001	0.0000
0.0000	0.0000
0.0000	0.0000

Next we consider Galerkin methods. We have observed in section 3, that the kernel of the integral operator is closely related to Laguerre polynomials. It is for this reason here we implement the projection scheme taking the subspace generated by the Laguerre polynomials.

5. Galerkin Method with Laguerre Polynomials

In this section, we take the target space as $L^2(0,\infty)$ with Laguerre polynomials as its basis. The Laguerre polynomials are generated using the recurrence relation

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= 1-x \\ kL_k(x) &= (2(k-1)+1-x)L_{k-1}-(k-1)L_{k-2}. \end{aligned}$$

Consider the Laguerre base functions $L_i(s), i = 0, \cdots N$. We assume that $g_N(s) = \sum_{i=0}^{N} c_i L_i(s)$ is an approximate solution of

$$u(x) = f(x) + \lambda \int_0^\infty k(x,t)u(t)dt, \quad 0 \le x,t < \infty. \tag{25}$$

Substituting in the integral equation (25), we get

$$\sum_{j=0}^{N} c_j L_j(s) = f(s) + \lambda \int_0^\infty k(s,t) \left(\sum_{i=0}^{N} c_i L_i(s) \right) dt.$$

Using orthogonality relation of Laguerre polynomials, it reduces to a system of linear equations

$$c_i - \lambda \sum_{j=0}^{N} c_j \langle h_j(s), L_i(s) \rangle = \langle 0, L_i(s) \rangle \qquad i = 0 \cdots N,$$

where $h_j(s) = \int_0^\infty k(s,t)L_j(t)dt$. From the solution $c_1, c_2, \cdots c_N$ of the above system, we can calculate $g_N(s)$. In Algorithm 3, we use the above idea to find the solution of (1).

Algorithm 3 Galerkin-Laguerre Method

Require: $(Tf)(x) = \int_0^1 \frac{(1-x)^{n+1}(n+tx)}{(1-tx)^{n+2}} t^{n-1} f(t)dt, n \in \mathbb{Z}, n \geq 1.$

Ensure: Solution of $Tf = f$

Change the limits of integration to $(0, \infty)$ by appropriate substitution

for i=1 to N **do**

  for j=1 to N **do**

    if i=j **then**

      $a_{ij} = 1 - \langle h_j(s), L_i(s) \rangle$

    else

      $a_{ij} = -\langle h_j(s), L_i(s) \rangle$

    end if

  end for

end for

Solve for $c = \{c_i, i = 1 \cdots N\}$the system $Ac^T = 0$, where $A = (a_{ij})_{1 \leq i,j \leq N}$

Construct $g_N(s) = \sum_{i=0}^{N} c_i L_i(s)$.

But this method does not work in finding solutions of (1) as c turns out to be the zero solution. The system can be perturbed by a small constant $\varepsilon > 0$ and can be solved.

6. Legendre-Galerkin method

In this method we choose the Legendre polynomials $\{\psi_0, \psi_1 \cdots \psi_N\}$ as an orthonormal basis for the target subspace X_N, where

$$\psi_0(x) = 1, \qquad \psi_1(x) = 1, x \in [-1, 1]$$

and for $i = 1, 2, 3 \cdots N-1$,

$$(i+1)\psi_{i+1}(x) = (2i+1)x\psi_i(x) - i\psi_{i-1}(x), \quad x \in [-1, 1].$$

Following similar procedure as explained for Laguerre polynomials, we get the following system of linear equations

$$c_i - \sum_{j=0}^{N} c_j \langle h_j(s), \psi_i(s) \rangle = \langle 0, \psi_i(s) \rangle \qquad i = 0 \cdots N.$$

The solution of this system would give the coefficients $c_i, i = 1 \cdots N$ in the construction of $u_N(s) = \sum_{i=0}^{N} c_i \psi_i(s)$, the approximate solution. Denoting $h_j(s) = \int_0^\infty k(s,t)\psi_j(t)dt$, we have the following algorithm (see Algorithm 4).

Algorithm 4 Legendre-Galerkin Method

Require: $(Tf)(x) = \int_0^1 \frac{(1-x)^{n+1}(n+tx)}{(1-tx)^{n+2}} t^{n-1} f(t)dt, n \in \mathbb{Z}, n \geq 1.$
Ensure: Solution of $Tf = f$

Change the limits of integral to $(-1,1)$ by appropriate substitution
fori=1 to N **do**
  for j=1 to N **do**
    if i=j **then**
      $a_{ij} = 1 - \langle h_j(s), \psi_i(s) \rangle$
    else
      $a_{ij} = -\langle h_j(s), \psi_i(s) \rangle$
    end if
  end for
end for
Solve for $c = \{c_i, i = 1 \cdots N\}$,the system $Ac^T = 0$, where $A = (a_{ij})_{1 \leq i,j \leq N}$
Compute $u_N(s) = \sum_{i=0}^{N} c_i \psi_i(s)$.

This method fails to solve the integral equations. Solutions of the perturbed system can be computed and analyzed.

7. Nystrom method for eigenvalue problem of integral operators

In this section we try the Nystrom method to solve the integral equation (1). We use Gauss-Legendre quadrature rule for approximating the integral. Nystrom method converts
$(Tf)(x) = \int_0^1 \frac{(1-x)^{n+1}(n+tx)}{(1-tx)^{n+2}} t^{n-1} f(t)dt = f(x), n \in \mathbb{Z}, n \geq 1$ to an algebraic eigenvalue problem $Ay = y$. We then find the eigenvalues and eigenvectors of A, which approximates the same of T. Here we shall look at the multiplicity of eigenvalue 1 of T.

Algorithm 5 Nystrom Method

Require: $(Tf)(x) = \int_0^1 \frac{(1-x)^{n+1}(n+tx)}{(1-tx)^{n+2}} t^{n-1} f(t)dt, n \in \mathbb{Z}, n \geq 1.$
Ensure: Solution of the integral equations $Tf = f$

Compute the weights $wt(i)$ $i = 1 \cdots N$ and node points $qpts(i)$ $i = 1 \cdots N$ of N-point Gaussian quadrature rule
  for i=1 to N **do**
    for j=1 to N **do**
      $a_{ij} = K(qpts(i), qpts(j))wt(j)$

TABLE 3. Eigenvalues of T

n=11	n=12
0.078239	0.383211
0.037221	0.092104
0.023457	0.026601
0.012072	0.007457
0.005027	4.695173

end for

end for

Compute the eigenvalues of A, where $A = (a_{ij})_{1\le i,j,\le N}$.

The eigenvalues obtained for $n = 11$ and $n = 12$ are listed in Table 3. But the method even fails to recognize the eigenvalue 1 in any case.

8. An algorithm for numerical solutions of integral equations associated with $\mathscr{M}$-harmonic functions

In this section, we present an algorithm for the solutions of the integral equations (1). The standard methods for solving integral equations don't seem to work for the integral equation (1). Only projection methods are giving some partial results. One can obtain the exact results using this projection methods provided unlimited computing resources are available. At this point, the Algorithm 6 proposed in this section is the best available method for solving the integral equations (1). We start with an introduction to hypergeometric series. Let $\Gamma(s)$ stands for the usual Gamma function, which is an analytic function of s in the whole complex plane except for simple poles at the points $\{0,-1,-2,\cdots\}$. In fact

$$\Gamma(z) = \frac{e^{-\beta z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1} e^{\frac{z}{n}}$$

where β is the Euler's constant; its approximate value is 0.57722. Hypergeometric series [7] is frequently used in connection with the theory of spherical harmonics. Let us consider $\Gamma(z)$, the gamma function. Then the Pochhammer symbol $(z)_n$ is defined by

$$(z)_n = \begin{cases} \dfrac{\Gamma(z+n)}{\Gamma(z)}, \text{ for } n = 0,1,2,\cdots; \\ \dfrac{(-1)^{|n|}}{(1-z)_{|n|}}, \text{for } n = -1,-2,\cdots. \end{cases}$$

The Gauss hypergeometric function $F(\alpha,\beta;\gamma;z)$ given by the power series

$$F(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}z^n \tag{26}$$

converges absolutely in the open unit disk $\mathbb{D}$ in $\mathbb{C}$ for all complex parameters α,β and γ with $\gamma \neq 0,-1,-2,\cdots$ and it is a particular solution of the hypergeometric equation

$$z(1-z)u'' + [\gamma-(\alpha+\beta+1)z]u' - \alpha\beta u = 0.$$

Since $(1-z)^{-(n+2)} = F(n+2,1;1;z)$ for $|z|<1$, the integral equation (1) is equivalent to

$$g(x) = (1-x)^{n+1}\int_0^1 (n+tx)F(n+2,1,1,tx)g(t)t^{n-1}dt. \tag{27}$$

We consider a more general form of the integral equation letting γ run over reals and $\gamma \geq 1$ instead of just positive integers. In this section we consider integral equations of the type

$$g(x) = (1-x)^{\gamma+1}\int_0^1 \frac{\gamma+tx}{(1-tx)^{\gamma+2}}g(t)t^{\gamma-1}dt, \quad \gamma\in\mathbb{R}, \gamma\geq 1. \tag{28}$$

These equations are Fredholm integral equations of second kind with the kernel function

$$\mathscr{K}(x,t) = (1-x)^{\gamma+1}\frac{\gamma+tx}{(1-tx)^{\gamma+2}}t^{\gamma-1}. \tag{29}$$

We can recast the integral equation in (28) as a fixed point problem $(T_\gamma g)(x) = g(x)$ where $(T_\gamma g)(x) = \int_0^1 \mathscr{K}(x,t)g(t)dt$. We present an algorithm for the solutions of the integral equation (28). The Algorithm 6 is developed from Yi's work [10] and we find the fixed points of the corresponding integral operators.

Algorithm 6 Solutions of integral equations associated with $\mathscr{M}$-harmonic functions

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Require: $\gamma\in\mathbb{R}, \gamma\geq 1$; $(T_\gamma g)(x) = (1-x)^{\gamma+1}\int_0^1 \frac{\gamma+tx}{(1-tx)^{\gamma+2}}g(t)t^{\gamma-1}dt$;

Ensure: f is a fixed point of T_γ

Compute $\Phi_\gamma(\beta) = \frac{\Gamma(\beta+1)\Gamma(\gamma+1-\beta)}{\Gamma(\gamma+1)}, \quad -1 < Re\beta < \gamma+1$

if $\gamma\in\mathbb{N}$ **then**

  if $z\in\mathbb{Z}$ **then**

   The solutions of $\Phi_\gamma(z) = 1$ are $z = 0$ and γ only.

  else

   Solve for $z\in\mathbb{C}\setminus\mathbb{Z}$, such that $\pi z\prod_{i=1}^{\gamma}(z-i) = (-1)^\gamma\Gamma(\gamma+1)sin(\pi z)$

  end if

else

  solve for $z\in\mathbb{C}$ such that $\Phi_\gamma(z) = 1$

end if

Compute $\lambda = -4z(\gamma-z)$

Verify $G_\gamma(\lambda) = \prod_{j=1}^{\infty}\left(1-\frac{\lambda}{4j(\gamma+j)}\right) = 1$

$F(z,z,\gamma,x) = 0$

for k=0 to L **do**

$$F(z,z,\gamma,x) = F(z,z,\gamma,x) + \frac{\left(\frac{\Gamma(z+k)}{\Gamma(z)}\right)^2}{\frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}}\frac{x^k}{k!}$$

end for
Compute $f(x) = c(1-x)^z F(z,z,\gamma,x)$ where c is an arbitrary constant, and $x \in [0,1)$.

Notice that in Algorithm 6 if $1 \le n \le 11, n \in \mathbb{Z}, \gamma = n$ then $z = 0$ or n. Suppose $z = n$. Then $f(x) = c(1-x)^n(1-x)^{-n} = c$ since $F(n,n,n,x) = (1-x)^{-n}$. Similarly if $z = 0$, $f(x) = c$.

The **if-else** part in the Algorithm 6 is tricky, which asks for complex solutions of the equation $\Phi_\gamma(z) - 1 = 0$. It is easy to check that when $\gamma \in \mathbb{N}$ and $z \in \mathbb{Z}$, $\Phi_\gamma(z) = 1$ only when $z = 0$ and γ. For $z \in \mathbb{C} \setminus \mathbb{Z}$, the expression for $\Phi_\gamma(z)$, $\gamma \in \mathbb{N}$ can be reformulated as the product of linear factors, $z, (z-1), (z-2)\cdots$. This form of the equation can be solved using standard numerical schemes like generalized Newton-Raphson method [2]. We illustrate the formulation for $\gamma = 1$ and the general form can be obtained by following similar procedures. Taking $\gamma = 1$ and substituting in $\Phi_\gamma(z) - 1 = 0$ we get

$$\Gamma(z+1)\Gamma(2-z) = 1. \tag{30}$$

From Euler's reflection formula, for $z \in \mathbb{C} \setminus \mathbb{Z}$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

and replacing z by $z-1$ we get

$$\Gamma(z-1)\Gamma(2-z) = \frac{\pi}{\sin(\pi(z-1))}.$$

Also $\Gamma(1+z) = z(z-1)\Gamma(z-1)$. Substituting in (30) we get,

$$z(1-z)\pi = \sin(\pi z).$$

Equating real and imaginary parts we obtain the system of equations

$$\pi(x - x^2 + y^2) - \cosh(\pi y)\sin(\pi x) = 0;$$
$$\pi(y - 2xy) - \sinh(\pi y)cos(\pi x) = 0.$$

The above system can be solved by using generalized Newton-Raphson method. The solutions are listed in Table 4. Also $z(1-z)\pi = \sin(\pi z)$ can be solved directly in terms of z using symbolic computation tools of MATLAB/MAPLE. Here one may observe that for $n = 1$ the solutions z that satisfy the condition $-1 < Rez < n+1$ are only 0 and 1. Thus the solutions computed are in conformation to the theoretical

TABLE 4. Values of z for different n

n=1	n=11	n=12
-2.21254007 - 1.2785689i	1.000000 + 0.000001i	11.99998 + 0.00002i
-4.29121059+1.6141967i	0.0000000	12.94279 + 1.86598i
0.00000000+0.00000000i	1.0000000	12.94277 + 1.85432i
1.00000000+0.00000000i	10.999996+0.000030i	11.999996543
1.00000000+0.00000006i	12.0024538 + 1.87947i	12.000001
5.29121059+ 1.6141967i	10.9999714 - 0.0000103i	12.000005 + 0.000002i
1.00000000-0.00000006i	11.000082 + 0.0000032i	0.00000000
3.21254007+ 1.27185689i	11.000066 + 0.0000003i	12.0000145 + 0.000001i

result obtained by Yi [10]. By following the same procedure one can obtain the values of z for different values of n.

9. An algorithm for numerical solutions of integral equations associated with hyperbolic-harmonic Functions

We next give an algorithm for computing the fixed points of the integral operator associated with the invariant-mean-value property of hyperbolically-harmonic functions.

Algorithm 7 Solutions of integral equations associated with hyperbolic-harmonic functions

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Require: $\gamma \in \mathbb{R}, \gamma \geq 2, (Sg)(x) = (1-x)^{\gamma}\frac{\gamma}{2}\int_0^1 \frac{1+tx}{(1-tx)^{\gamma+1}} g(t)t^{\frac{\gamma}{2}-1}dt.$

Ensure: f is a fixed point of S.

Compute $\Psi_{\gamma}(\beta) = \frac{\Gamma(\beta+1)\Gamma(\gamma-\beta)}{\Gamma(\gamma)}, \quad -1 < Re\beta < \gamma.$

if $\gamma \in \mathbb{N}$ **then**

if $z \in \mathbb{Z}$ **then**

The solutions of $\Psi_{\gamma}(z) = 1$ are $z = 0$ and $\gamma - 1$ only

else

solve for $z \in \mathbb{C}$ such that $\pi z \prod_{i=1}^{\gamma-1}(z-i) = (-1)^{\gamma-1}\Gamma(\gamma)sin(\pi z)$

end if

else

solve for $z \in \mathbb{C}$ such that $\Psi_{\gamma}(z) = 1$

end if

Find $\lambda = -4z(\gamma-1-z)$.

Verify $G_{\gamma}(\lambda) = \prod_{j=1}^{\infty}\left(1 - \frac{\lambda}{4j(\gamma-1+j)}\right) = 1$

$F(z, z-\gamma/2+1, \gamma/2, x) = 0$

for k=0 to L **do**

$$F(z, z-\gamma/2+1, \gamma/2, x) = F(z, z-\gamma/2+1, \gamma/2, x) + \frac{\frac{\Gamma(z+k)}{\Gamma(z)}\frac{\Gamma(z-\gamma/2+1+k)}{\Gamma(z-\gamma/2+1)}}{\frac{\Gamma(\gamma/2+k)}{\Gamma(\gamma/2)}}\frac{x^k}{k!}$$

end for

Find $f(x) = c(1-x)^z F(z, z-\frac{\gamma}{2}+1, \frac{\gamma}{2}, x), x \in [0,1)$ where c is an arbitrary constant.

If $2 \leq \gamma \leq 12$, in Algorithm 7 then from [8], it follows that $z = 0$ or $\gamma - 1$. In that case, $f \equiv c$ as $F(\gamma-1, \frac{\gamma}{2}, \frac{\gamma}{2}, x) = (1-x)^{1-\gamma}$.

The formulation in **if-else** part used in Algorithm 7 for $\gamma \in \mathbb{N}$ is same as in the case of Algorithm 6,which can be easily solved numerically. We illustrate the formulation for $\gamma = 3$. Taking $\gamma = 3$ and substituting in $\Psi_{\gamma}(z) - 1 = 0$ we get

$$\Gamma(z+1)\Gamma(3-z) = \Gamma(3). \tag{31}$$

Using, Euler's reflection formula, for $z \in \mathbb{C} \setminus \mathbb{Z}$,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

and replacing z by $z-2$ we get

$$\Gamma(z-2)\Gamma(3-z) = \frac{\pi}{\sin(\pi(z-2))}.$$

Also $\Gamma(1+z) = z(z-1)(z-2)\Gamma(z-2)$. Substituting in (31) we get,

$$\pi z(z-1)(z-2) = \Gamma(3)\sin(\pi z).$$

Equating real and imaginary parts we obtain the system of equations

$$\pi(x^3 - 3xy^2 + 3y^2 - 3x^2 + 2x) - \cosh(\pi y)\sin(\pi x) = 0;$$

$$\pi(3x^2y - 6xy - y^3 + 2y) - \sinh(\pi y)cos(\pi x) = 0.$$

The above system can be solved by using generalized Newton-Raphson method [2]. The solutions are listed in table 5. Also $\pi z(z-1)(z-2) = \Gamma(3)\sin(\pi z)$ can be solved directly in terms of z using symbolic computation tools of MATLAB/MAPLE. Here one may observe that for $n = 3$ the solutions z that satisfy the condition $-1 < Rez < n+1$ are only 0 and 2. The solutions obtained are in conformation with the theoretical results obtained by Jevtic [8].

TABLE 5. Values of z for different n

n=3	n=12	n=13
-0.0016512 + 0.00164732i	1.000000 + 0.000001i	11.99998 + 0.00002i
-0.0015396 + 0.0015362i	0.0000000	12.94279 + 1.86598i
0.0023145 + 0.00023068i	1.0000000	12.94277 + 1.85432i
2.00172405 + 0.001719i	10.999996+0.000030i	11.999996543
2.00161288 + 0.0001609i	12.0024538 + 1.87947i	12.000001
4.00489658 + 1.3926428i	10.9999714 - 0.0000103i	12.000005 + 0.000002i
6.12710495 + 1.8643223i	11.000082 + 0.0000032i	0.00000000
0.0001615 + 0.00013472i	11.000066 + 0.0000003i	12.0000145 + 0.000001i

More generally, one can consider the integral equation

$$T_{s,\gamma}f(x) = \int_0^1 K_{s,\gamma}(x,t)f(t)t^{\gamma-1}dt, s = 0,1,2,\cdots \tag{32}$$

for $f \in L^1((0,1),x^{\gamma-1}dx)$ where the kernel

$$K_{s,\gamma}(x,t) = \frac{\Gamma(\gamma+s+1)}{\Gamma(\gamma)\Gamma(s+1)}(1-x)^{\gamma+s+1}F(\gamma+s+1,\gamma+s+1;\gamma;tx)(1-t)^s$$

and $F(\gamma+s+1,\gamma+s+1;\gamma;tx)$ is the Gauss hypergeometric function. One can verify that $T_{s,\gamma}$ is real analytic for all $s \in \mathbb{Z}_+$. When $s = 0$, the Euler transformation [10] shows

$$K_{0,\gamma}(x,t) = \frac{(1-x)^{\gamma+1}}{(1-tx)^{\gamma+2}}(\gamma+tx)$$

and hence

$$T_{0,\gamma}f(x) = (1-x)^{\gamma+1}\int_0^1 \frac{(\gamma+tx)}{(1-tx)^{\gamma+2}} f(t)t^{\gamma-1}dt \tag{33}$$

which is a continuous parameter version of (1).

References

[1] Ahern, P., Flores, M., Rudin, W., ”An invariant volume-mean-value property”, *J.Funct. Anal.* 111, 380-397, 1993.
[2] Atkinson, K. E., An Introduction to Numerical Analysis, 2nd ed., *John Wiley*, New York, 1988.
[3] Axler, S., Cuckovic, Z., ”Commuting Toeplitz operators with harmonic symbols”, *Integral Equations Operator Theory* 14, 1-12, 1991.
[4] Chatelin, F., *“Spectral Approximation of Linear Operators”*, Academic Press, 1983.
[5] Englis, M., “Functions invariant under the Berezin transform”, J. Funct. Anal. 121, 233-254, 1994.
[6] Englis, M., Toeplitz operators on Bergman-type spaces, Thesis, Praha, 1991.
[7] Gelfand, I. M., Spherical functions on symmetric Riemannian spaces, Dokl. Akad. Nauk SSSR 70, 5–8(Russian), 1950.
[8] Jevtic, M., “Fixed points of an integral operator”, J. Anal. Math. 91, 123–141, 2003.
[9] Rudin, W., Functional Analysis, 2nd Edition, McGraw-Hill, New York, 1991.
[10] Yi, J.S., A characterization of the functions fixed by a class of integral operators, Ph.D. thesis, Univ. of Wisconsin, Madison, 1995.