



## The multi Rough Ideal Convergence of Difference Strongly of $\chi^2$ in $p$ - Metric spaces Defined by Orlicz

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**Abstract.** The aim of this paper is to introduce multi rough and study a new concept of the  $\chi^2$  space via ideal convergence of difference operator defined by Orlicz. Some topological properties of the resulting sequence spaces are also discussed.

**Keywords.** Analytic sequence, modulus function, double sequences,  $\chi^2$  space,  $p$ -metric space, multi rough ideal.

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### 1. Introduction

The idea of rough convergence was introduced by Phu [3], who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal et al. [2] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence. Statistical convergence and ideal convergence was introduced by Mursaleen [7] and Tripathy et al. [8-13] and many others. In this paper we have introduced some Orlicz sequence spaces of fuzzy number using the notion of rough I- convergence and studied some algebraic and topological properties of these spaces.

Throughout  $w, \chi$  and  $\Lambda$  denote the classes of all, gai and analytic scalar valued single sequences respectively. We write  $w^2$  for the set of all complex double sequences  $(x_{mn})$ , where  $m, n \in \mathbb{N}$ , the set of positive integers. Then,  $w^2$  is a linear space under the coordinate wise addition and scalar multiplication. Let  $(x_{mn})$  be a double sequence of real or complex numbers. Then the series  $\sum_{m,n=1}^{\infty} x_{mn}$  is called a double series. The double series  $\sum_{m,n=1}^{\infty} x_{mn}$  give one space is said to be convergent if and only if the double sequence  $(S_{mn})$  is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij}(m, n = 1, 2, 3, \dots).$$

A double sequence  $x = (x_{mn})$  is said to be double analytic if

$$\sup_{m,n} |x_{mn}|^{\frac{1}{m+n}} < \infty.$$

The vector space of all double analytic sequences are usually denoted by  $\Lambda^2$ . A sequence  $x = (x_{mn})$  is called double entire sequence if

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$$|x_{mn}|^{\frac{1}{m+n}} \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

The vector space of all double entire sequences are usually denoted by  $\Gamma^2$ . Let the set of sequences with this property be denoted by  $\Lambda^2$  and  $\Gamma^2$  is a metric space with the metric

$$d(x, y) = \sup_{m, n} \left\{ |x_{mn} - y_{mn}|^{\frac{1}{m+n}} : m, n : 1, 2, 3, \dots \right\}, \quad (1)$$

for all  $x = \{x_{mn}\}$  and  $y = \{y_{mn}\}$  in  $\Gamma^2$ . Let  $\phi = \{\text{finite sequences}\}$ .

Consider a double sequence  $x = (x_{mn})$ . The  $(m, n)^{th}$  section  $x^{[m, n]}$  of the sequence is defined by  $x^{[m, n]} = \sum_{i, j=0}^{m, n} x_{ij} \delta_{ij}$  for all  $m, n \in \mathbb{N}$ ,

$$\delta_{mn} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots \\ \vdots & & & & & \\ 0 & 0 & \dots & 1 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots \end{pmatrix}$$

with 1 in the  $(m, n)^{th}$  position and zero otherwise.

An Orlicz function is a function  $f : [0, \infty) \rightarrow [0, \infty)$  which is continuous, non-decreasing and convex with  $f(0) = 0$ ,  $f(x) > 0$ , for  $x > 0$  and  $f(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . If convexity of Orlicz function  $f$  is replaced by  $f(x+y) \leq f(x) + f(y)$ , then this function is called modulus function. An modulus function  $f$  is said to satisfy  $\Delta^2$ - condition for all values  $u$ , if there exists  $K > 0$  such that  $f(2u) \leq Kf(u)$ ,  $u \geq 0$ .

**Remark 1:** An Modulus function satisfies the inequality  $f(\lambda x) \leq \lambda f(x)$  for all  $\lambda$  with  $0 < \lambda < 1$ .

### 1.1. Lemma

Let  $f$  be an modulus function which satisfies  $\Delta^2$ - condition and let  $0 < \delta < 1$ . Then for each  $t \geq \delta$ , we have  $f(t) < K\delta^{-1}f(2)$  for some constant  $K > 0$ .

Let  $M$  and  $\Phi$  be mutually complementary modulus functions. Then, we have

(i) For all  $u, y \geq 0$ ,

$$uy \leq M(u) + \Phi(y), (\text{Young's inequality}), [\text{see}[4]]. \quad (2)$$

(ii) For all  $u \geq 0$ ,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \quad (3)$$

(iii) For all  $u \geq 0$ , and  $0 < \lambda < 1$ ,

$$M(\lambda u) \leq \lambda M(u). \quad (4)$$

[5] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

The space  $\ell_M$  with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For  $M(t) = t^p$  ( $1 \leq p < \infty$ ), the spaces  $\ell_M$  coincide with the classical sequence space  $\ell_p$ .

A sequence  $f = (f_{mn})$  of modulus function is called a Musielak-modulus function. A sequence  $g = (g_{mn})$  defined by

$$g_{mn}(v) = \sup \{ |v|u - f_{mn}(u) : u \geq 0 \}, m, n = 1, 2, \dots$$

is called the complementary function of a Musielak-modulus function  $f$ . For a given Musielak modulus function  $f$ , the Musielak-modulus sequence space  $t_f$  is defined by

$$t_f = \left\{ x \in w^3 : M_f(|x_{mnk}|)^{1/m+n+k} \rightarrow 0 \text{ as } m, n, k \rightarrow \infty \right\},$$

where  $M_f$  is a convex modular defined by

$$M_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}(|x_{mnk}|)^{1/m+n+k}, x = (x_{mnk}) \in t_f.$$

We consider  $t_f$  equipped with the Luxemburg metric space.

Let  $(X_i, d_i), i \in I$  be a family of metric spaces such that each two elements of the family are disjoint. Denote  $X : \bigcup_{i \in I} X_i$ . If we define

$$d(x, y) = \begin{cases} d_i(x, y), & \text{if } x, y \in X_i \\ +\infty & \text{if } x \in X_i, y \in X_j, i \neq j \end{cases}$$

then the pair  $(X, d)$  is a Luxemburg metric space. The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [6] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\},$$

for  $Z = c, c_0$  and  $\ell_\infty$ , where  $\Delta x_k = x_k - x_{k+1}$  for all  $k \in \mathbb{N}$ .

Here  $c, c_0$  and  $\ell_\infty$  denote the classes of convergent, null and bounded scalar valued single sequences respectively. The difference sequence space  $bv_p$  of the classical space  $\ell_p$  is introduced and studied in the case  $1 \leq p \leq \infty$  by Başar and Altay and in the case  $0 < p < 1$ . The spaces  $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$  and  $bv_p$  are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = (\sum_{k=1}^{\infty} |x_k|^p)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\},$$

where  $Z = \Lambda^2, \chi^2$  and  $\Delta x_{mn} = (x_{mn} - x_{m+1n}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{m+1n} - x_{m+1n} + x_{m+1n+1}$  for all  $m, n \in \mathbb{N}$ . The generalized difference double notion has the following representation:  $\Delta^m x_{mn} = \Delta^{m-1} x_{mn} - \Delta^{m-1} x_{m+1n} - \Delta^{m-1} x_{m+1n+1} + \Delta^{m-1} x_{m+1n+1}$ , and also this generalized difference double notion has the following binomial representation:  $\Delta^m x_{mn} = \sum_{i=0}^m \sum_{j=0}^m (-1)^{i+j} \binom{m}{i} \binom{m}{j} x_{m+i, n+j}$ .

## 2. Definitions and Preliminaries

Let  $\Delta^m X$  be a non empty set. A non-void class  $I \subseteq 2^{\Delta^m X}$  (power set, of  $\Delta^m X$ ) is called an ideal if  $I$  is additive (i.e  $A, B \in I \Rightarrow A \cup B \in I$ ) and hereditary (i.e  $A \in I$  and  $B \subseteq A \Rightarrow B \in I$ ). A non-empty family of sets  $F \subseteq 2^{\Delta^m X}$  is said to be a filter on  $\Delta^m X$  if  $\emptyset \notin F; A, B \in F \Rightarrow A \cap B \in F$  and  $A \in F, A \subseteq B \Rightarrow B \in F$ . For each ideal  $I$  there is a filter  $F(I)$  given by  $F(I) = \{K \subseteq N : N \setminus K \in I\}$ . A non-trivial ideal  $I \subset 2^{\Delta^m X}$  is called admissible if and only if  $\{\{x\} : x \in \Delta^m X\} \subset I$ .

A double sequence space  $E$  is said to be solid or normal if  $(\alpha_{mn} \Delta^m x_{mn}) \in E$ , whenever  $(\Delta^m x_{mn}) \in E$  and for all double sequences  $\alpha = (\alpha_{mn})$  of scalars with  $|\alpha_{mn}| \leq 1$ . for all  $m, n \in \mathbb{N}$ .

Let  $n \in \mathbb{N}$  and  $X$  be a real vector space of dimension  $w$ , where  $n \leq w$ . A real valued function  $d_p(x_1, \dots, x_n) = \|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p$  on  $X$  satisfying the following four conditions:

(i)  $\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p = 0$  if and only if  $d_1(x_1, 0), \dots, d_n(x_n, 0)$  are linearly dependent,

(ii)  $\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p$  is invariant under permutation,

(iii)  $\|(\alpha d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p = |\alpha| \|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p, \alpha \in \mathbb{R}$

(iv)  $d_p((x_1, y_1), (x_2, y_2) \dots (x_n, y_n)) = (d_X(x_1, x_2, \dots, x_n)^p + d_Y(y_1, y_2, \dots, y_n)^p)^{1/p}$  for  $1 \leq p < \infty$ ; (or)

(v)  $d((x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)) := \sup \{d_X(x_1, x_2, \dots, x_n), d_Y(y_1, y_2, \dots, y_n)\}$ ,

for  $x_1, x_2, \dots, x_n \in X, y_1, y_2, \dots, y_n \in Y$  is called the  $p$ -product metric of the Cartesian product of  $n$ -metric spaces is the  $p$ -norm of the  $n$ -vector of the norms of the  $n$ -sub spaces.

A trivial example of  $p$ -product metric of  $n$ -metric space is the  $p$ -norm space is  $X = \mathbb{R}$  equipped with the following Euclidean metric in the product space is the  $p$ -norm:

$$\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_E = \sup(|\det(d_{mn}(x_{mn}, 0))|) = \sup \left( \begin{array}{cccc} d_{11}(x_{11}, 0) & d_{12}(x_{12}, 0) & \dots & d_{1n}(x_{1n}, 0) \\ d_{21}(x_{21}, 0) & d_{22}(x_{22}, 0) & \dots & d_{2n}(x_{2n}, 0) \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1}(x_{n1}, 0) & d_{n2}(x_{n2}, 0) & \dots & d_{nn}(x_{nn}, 0) \end{array} \right)$$

where  $x_i = (x_{i1}, \dots, x_{in}) \in \mathbb{R}^n$  for each  $i = 1, 2, \dots, n$ .

If every Cauchy sequence in  $X$  converges to some  $L \in X$ , then  $X$  is said to be complete with respect to the  $p$ - metric. Any complete  $p$ - metric space is said to be  $p$ - Banach metric space.

### 2.1. Defintion

A sequence  $(x_{mn})$  is said to be rough convergent ( $r$ -convergent) to  $\bar{0}$  if for every  $\varepsilon > 0$  there exists a positive integer  $n_0$  such that  $\|x_{mn}, \bar{0}\| < r + \varepsilon$  for all  $n > n_0$ , where  $r$  is a non negative real number called the convergence degree.

### 2.2. Definition

A sequence  $(x_{mn})$  is said to be rough  $I$ - convergent to  $\bar{0}$  if for each  $\varepsilon > 0$ ,  $\{m, n \in \mathbb{N} : \|x_{mn}, \bar{0}\| > r + \varepsilon\} \in I$ . Here  $\bar{0}$  is called the rough  $I$ - limit of the sequence  $(x_{mn})$  and we write  $I_r - \lim x_{mn} = \bar{0}$ .

### 3. Main Results

In this section we introduce the notion of different types of  $I$ - convergent double sequences. This generalizes and unifies different notions of convergence for  $\chi^2$ . We shall denote the ideal of  $2^{N \times N}$  by  $I_2$ .

Let  $I_2$  be an ideal of  $2^{N \times N}$ ,  $f$  be an modulus function. Let  $u$  and  $v$  be two non-negative integers and  $\mu = (\mu_{mn})$  be a sequence of non-zero reals. Then for a sequence  $\eta = (\eta_{mn})$  be a double analytic sequence of strictly positive real numbers and  $(\Delta_{(\mu,u)}^v X, \|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p)$  be an  $p$ - product of  $n$  metric spaces is the  $p$  norm of the  $n$ -vector of the norms of the  $n$  subspaces. Further  $\chi^2(p - \Delta_{(\mu,u)}^v X)$  denotes  $\Delta_{(\mu,u)}^v X$ - valued sequence space. Now, we define the following sequence spaces:

$$\begin{aligned} \chi_{\Delta_{(\mu,u)}^v}^{2I_2, f} [\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p]^\eta = x = (\Delta_{(\mu,u)}^v x_{mn}) \in \chi^2(p - \Delta_{(\mu,u)}^v X) : \forall \varepsilon > 0, \\ (r, s) \in N \times N : \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left( \left| \Delta_{(\mu,u)}^v x_{mn} \right|^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \right]^{\eta_{mn}} \geq r + \varepsilon \\ \in I_2, \text{ for every } d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \in \Delta_{(\mu,u)}^v X. \end{aligned}$$

$$\begin{aligned} \Lambda_{\Delta_{(\mu,u)}^v}^{2I_2, f} [\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p]^\eta = x = (x_{mn}) \in \Lambda^2(p - \Delta_{(\mu,u)}^v X) : \exists K > 0, \\ (r, s) \in N \times N : \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left( \left| \Delta_{(\mu,u)}^v x_{mn} \right|^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \right]^{\eta_{mn}} \geq r + K \in I_2, \text{ for every } \\ d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \in \Delta^m X. \end{aligned}$$

$$\begin{aligned} \Lambda_{\Delta_{(\mu,u)}^v}^2 [\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p]^\eta = x = (\Delta^m x_{mn}) \in \Lambda^2(p - \Delta_{(\mu,u)}^v X) : \exists K > 0, \\ (r, s) \in N \times N : \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left( \left| \Delta_{(\mu,u)}^v x_{mn} \right|^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \right]^{\eta_{mn}} \leq r + K, \text{ for every } \\ d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \in \Delta_{(\mu,u)}^v X. \end{aligned}$$

If  $\eta = \eta_{mn} = 1$  for all  $m, n \in \mathbb{N}$  we obtain

$$\chi_{\Delta_{(\mu,u)}^v}^{2I_2, f} [\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p]^\eta = \chi_{\Delta_{(\mu,u)}^v}^{2I_2, f} [\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p],$$

$$\Lambda_{\Delta_{(\mu,u)}^v}^{2I_2, f} [\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p]^\eta = \Lambda_{\Delta_{(\mu,u)}^v}^{2I_2, f} [\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p],$$

$$\Lambda_{\Delta_{(\mu,u)}^v}^2 [\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p]^\eta = \Lambda_{\Delta_{(\mu,u)}^v}^2 [\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p].$$

The following well-known inequality will be used in this study:  $0 \leq \inf_{mn} \eta_{mn} = H_0 \leq \eta_{mn} \leq \sup_{mn} \eta_{mn} = H < \infty, D = \max(1, 2^{H-1})$ , then

$$|x_{mn} + y_{mn}|^{\eta_{mn}} \leq D \{ |x_{mn}|^{\eta_{mn}} + |y_{mn}|^{\eta_{mn}} \}$$

for all  $m, n \in \mathbb{N}$  and  $x_{mn}, y_{mn} \in \mathbb{C}$ . Also  $|x_{mn}|^{\eta_{mn}/m+n} \leq \max(1, |x_{mn}|^{H/m+n})$  for all  $x_{mn} \in \mathbb{C}$ .

#### 3.1. Theorem

The classes of sequences  $\chi_{\Delta_{(\mu,u)}^v}^{2I_2, f} [\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p]^\eta$ ,

$\Lambda_{\Delta_{(\mu,u)}^v}^{2I_2, f} [\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p]^\eta$  are linear spaces over the complex field  $\mathbb{C}$

**Proof:** Now we establish the result for the case  $\chi_{\Delta_{(\mu,u)}^v}^{2I_2, f} [\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p]^\eta$  and the others can be proved similarly. Let  $x, y \in \chi_{\Delta_{(\mu,u)}^v}^{2I_2, f} [\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p]^\eta$  and  $\alpha, \beta \in \mathbb{C}$ . Then

$$\left\{ (r, s) \in N \times N : \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left\| \left( \left| \Delta_{(\mu, u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right]^{\eta_{mn}} \geq r + \frac{\varepsilon}{2} \right\} \in I_2 \text{ and}$$

$$\left\{ (r, s) \in N \times N : \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left\| \left( \left| \Delta_{(\mu, u)}^v y_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right]^{\eta_{mn}} \geq r + \frac{\varepsilon}{2} \right\} \in I_2.$$

Since  $\|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p$  be an  $p$ - product of  $n$  metric spaces is the  $p$  norm of the  $n$ -vector of the norms of the  $n$  subspaces and  $f$  is an Orlicz function, the following inequality holds:

$$\begin{aligned} & \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left( \left\| \frac{(\alpha \Delta_{(\mu, u)}^v x_{mn} + \beta \Delta_{(\mu, u)}^v y_{mn})^{1/m+n}}{|\alpha|^{1/m+n} + |\beta|^{1/m+n}}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right) \right]^{\eta_{mn}} \leq \\ & \frac{D}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ \frac{|\alpha|^{1/m+n}}{|\alpha|^{1/m+n} + |\beta|^{1/m+n}} f \left( \left\| \left( \left| \Delta_{(\mu, u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right) \right]^{\eta_{mn}} + \\ & \frac{D}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ \frac{|\beta|^{1/m+n}}{|\alpha|^{1/m+n} + |\beta|^{1/m+n}} f \left( \left\| \left( \left| \Delta_{(\mu, u)}^v y_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right) \right]^{\eta_{mn}} \leq \\ & \frac{D}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left( \left\| \left( \left| \Delta_{(\mu, u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right) \right]^{\eta_{mn}} + \\ & \frac{D}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left( \left\| \left( \left| \Delta_{(\mu, u)}^v y_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right) \right]^{\eta_{mn}}. \end{aligned}$$

From the above inequality we get

$(r, s) \in N \times N$ :

$$\begin{aligned} & \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left( \left\| \frac{(\alpha \Delta_{(\mu, u)}^v x_{mn} + \beta \Delta_{(\mu, u)}^v y_{mn})^{1/m+n}}{|\alpha|^{1/m+n} + |\beta|^{1/m+n}}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right) \right]^{\eta_{mn}} \\ & \geq r + \varepsilon \subset (r, s) \in N \times N : \\ & \frac{D}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left\| \left( \left| \Delta_{(\mu, u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right]^{\eta_{mn}} \\ & \geq r + \frac{\varepsilon}{2} \in I_2 \cup (r, s) \in N \times N : \\ & \frac{D}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left\| \left( \left| \Delta_{(\mu, u)}^v y_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right]^{\eta_{mn}} \geq r + \frac{\varepsilon}{2} \\ & \in I_2. \text{ This completes the proof.} \end{aligned}$$

### 3.2. Theorem

The class of sequence  $\chi_{\Delta_{(\mu, u)}^v f}^{2I_2} [ \|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p ]^\eta$  is a paranormed space with respect to the paranorm defined by

$$g_{rs}(x) = \inf \left\{ \left( \sup_{rs} \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left\| \left( \left| \Delta_{(\mu, u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1), \dots, d_n(x_{n-1}) \right\|_p \right]^{\eta_{mn}} \right)^{\frac{1}{H}} \leq r \right\}, \text{ for ev-}$$

ery  $d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \in X$ .

**Proof:**  $g_{rs}(\theta) = 0$  and  $g_{rs}(-x) = g_{rs}(x)$  are easy to prove, so we omit them. Let us take  $x, y \in \chi_{\Delta_{(\mu, u)}^v f}^{2I_2} [ \|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p ]^{\eta_{mn}}$ .

Let

$$g_{rs}(x) = \inf \left\{ \sup_{rs} \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left\| \left( \left| \Delta_{(\mu, u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right]^{\eta_{mn}} \leq r, \forall x \in X \right\}$$

and

$$g_{rs}(y) = \inf \left\{ \sup_{rs} \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left\| \left( \left| \Delta_{(\mu, u)}^v y_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right]^{\eta_{mn}} \leq r, \forall x \in X \right\}.$$

Then we have

$$\begin{aligned} & \sup_{rs} \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left\| \left( \left( \left| \Delta_{(\mu,u)}^v x_{mn} + \Delta_{(\mu,u)}^v y_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \right\|_p \right]^{\eta_{mn}} \leq \\ & \sup_{rs} \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left\| \left( \left( \left| \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \right\|_p \right]^{\eta_{mn}} + \\ & \sup_{rs} \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left\| \left( \left( \left| \Delta_{(\mu,u)}^v y_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \right\|_p \right]^{\eta_{mn}}. \end{aligned}$$

Thus

$$\sup_{rs} \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left\| \left( \left( \left| \Delta_{(\mu,u)}^v x_{mn} + \Delta_{(\mu,u)}^v y_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \right\|_p \right]^{\eta_{mn}} \leq 1 \text{ and } g_{rs}(x+y) = g_{rs}(x) + g_{rs}(y).$$

Now, let  $\lambda_{mn}^u \rightarrow \lambda$ , where  $\lambda_{mn}^u, \lambda \in \mathbb{C}$  and  $g_{rs}(\Delta_{(\mu,u)}^v x_{mn}^u - \Delta_{(\mu,u)}^v x_{mn}) \rightarrow 0$  as  $u \rightarrow \infty$ . We have to prove that  $g_{rs}(\lambda_{mn} \Delta_{(\mu,u)}^v x_{mn}^u - \lambda \Delta_{(\mu,u)}^v x_{mn}) \rightarrow 0$  as  $u \rightarrow \infty$ . Let

$$g_{rs}(x^u) = \left\{ \sup_{rs} \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left\| \left( \left( \left| \Delta_{(\mu,u)}^v x_{mn}^u \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \right\|_p \right]^{\eta_{mn}} \leq r, \forall x \in X \right\}$$

and

$$g_{rs}(x^u - x) = \left\{ \sup_{rs} \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left\| \left( \left( \left| \Delta_{(\mu,u)}^v x_{mn}^u - \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \right\|_p \right]^{\eta_{mn}} \leq r, \text{ for all } x \in X \right\}.$$

We observe that

$$\begin{aligned} & f \left( \left\| \left( \frac{\left( \left| \lambda_{mn}^u \Delta_{(\mu,u)}^v x_{mn}^u - \lambda \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}}{|\lambda_{mn}^u - \lambda|^{1/m+n} + |\lambda|^{1/m+n}}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \right\|_p \right) \leq \\ & f \left( \left\| \left( \frac{\left( \left| \lambda_{mn}^u x_{mn}^u - \lambda \Delta_{(\mu,u)}^v x_{mn}^u \right| \right)^{1/m+n}}{|\lambda_{mn}^u - \lambda|^{1/m+n} + |\lambda|^{1/m+n}}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \right\|_p \right) + \\ & f \left( \left\| \left( \frac{\left( \left| \lambda \Delta_{(\mu,u)}^v x_{mn}^u - \lambda \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}}{|\lambda_{mn}^u - \lambda|^{1/m+n} + |\lambda|^{1/m+n}}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \right\|_p \right) \leq \\ & \frac{|\lambda_{mn}^u - \lambda|}{|\lambda_{mn}^u - \lambda| + |\lambda|} f \left( \left\| \left( \left( \left| \Delta_{(\mu,u)}^v x_{mn}^u \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \right\|_p \right) + \\ & \frac{|\lambda|}{|\lambda_{mn}^u - \lambda| + |\lambda|} f \left( \left\| \left( \left( \left| \Delta_{(\mu,u)}^v x_{mn}^u - \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \right\|_p \right). \end{aligned}$$

From this inequality, it follows that

$$\left[ f \left( \left\| \left( \frac{\left( \left| \lambda_{mn}^u \Delta_{(\mu,u)}^v x_{mn}^u - \lambda \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}}{|\lambda_{mn}^u - \lambda|^{1/m+n} + |\lambda|^{1/m+n}}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \right\|_p \right) \right]^{\eta_{mn}} \leq r$$

and consequently

$$\begin{aligned} g_{rs}(\lambda_{mn}^u \Delta_{(\mu,u)}^v x_{mn}^u - \lambda \Delta_{(\mu,u)}^v x_{mn}) & \leq (|\lambda_{mn}^u - \lambda|)^{\frac{\eta_{mn}}{H}} \inf \left\{ g_{rs}(\Delta_{(\mu,u)}^v x_{mn}^u) \right\} + \\ & (|\lambda|)^{\frac{\eta_{mn}}{H}} \inf \left\{ g_{rs}(\Delta_{(\mu,u)}^v x_{mn}^u - \Delta_{(\mu,u)}^v x_{mn}) \right\} \leq \max \left\{ |\lambda|, (|\lambda|)^{\frac{\eta_{mn}}{H}} \right\} g_{rs}(\Delta_{(\mu,u)}^v x_{mn}^u - \Delta_{(\mu,u)}^v x_{mn}). \end{aligned}$$

Hence by our assumption the right hand side tends to zero as  $u, m$  and  $n \rightarrow \infty$ . This completes the proof.

### 3.3. Theorem

(i) If  $0 < inf_{mn} \eta_{mn} = H_0 \leq \eta_{mn} < 1$ , then

$$\chi_{\Delta_{(\mu,u)}^v}^{2I_2} [ \| (d_1(x_1, 0), \dots, d_n(x_n, 0)) \|_p ]^\eta \subset \chi_{\Delta_{(\mu,u)}^v}^{2I_2} [ \| (d_1(x_1, 0), \dots, d_n(x_n, 0)) \|_p ] .$$

(ii) If  $1 \leq \eta_{mn} \leq sup_{mn} \eta_{mn} = H < \infty$ , then  $\chi_{\Delta_{(\mu,u)}^v}^{2I_2} [ \| (d_1(x_1, 0), \dots, d_n(x_n, 0)) \|_p ] \subset \chi_f^{2I_2} [ \| (d_1(x_1, 0), \dots, d_n(x_n, 0)) \|_p ]^\eta$ .

(iii) If  $0 < \eta_{mn} < \mu_{mn} < \infty$  and  $\left\{ \frac{\mu_{mn}}{\eta_{mn}} \right\}$  is double analytic, then

$$\chi_{\Delta_{(\mu,u)}^v}^{2I_2} [ \| (d_1(x_1, 0), \dots, d_n(x_n, 0)) \|_p ]^\eta \subset \chi_{\Delta_{(\mu,u)}^v}^{2I_2} [ \| (d_1(x_1, 0), \dots, d_n(x_n, 0)) \|_p ]^\mu .$$

**Proof:** The proof can be established using standard technique.

The following result is well known.

### 3.4. Lemma

If a sequence space E is solid, then it is monotone.

### 3.5. Theorem

The class of sequence  $\chi_{\Delta_{(\mu,u)}^v}^{2I_2} [ \| (d_1(x_1, 0), \dots, d_n(x_n, 0)) \|_p ]^\eta$  is not solid and hence not monotone

**Proof:** It is routine verification. Therefore we omit the proof.

### 3.6. Theorem

Let  $f, f_1$  and  $f_2$  be Orlicz functions. Then we have

$$(i) \chi_{\Delta_{(\mu,u)}^v}^{2I_2} [ \| (d_1(x_1, 0), \dots, d_n(x_n, 0)) \|_p ]^\eta \subset \chi_{\Delta_{(\mu,u)}^v}^{2I_2} [ \| (d_1(x_1, 0), \dots, d_n(x_n, 0)) \|_p ]^\eta$$

$$(ii) \chi_{\Delta_{(\mu,u)}^v}^{2I_2} [ \| (d_1(x_1, 0), \dots, d_n(x_n, 0)) \|_p ]^\eta \cap \chi_{\Delta_{(\mu,u)}^v}^{2I_2} [ \| (d_1(x_1, 0), \dots, d_n(x_n, 0)) \|_p ]^\eta \subset$$

$$\chi_{\Delta_{(\mu,u)}^v}^{2I_2} [ \| (d_1(x_1, 0), \dots, d_n(x_n, 0)) \|_p ]^\eta$$

**Proof:** (i) Let  $inf_{mn} \eta_{mn} = H_0$ . For given  $r + \varepsilon > 0$ , we first choose  $r + \varepsilon_0 > 0$  such that  $max \left\{ r + \varepsilon_0^H, r + \varepsilon_0^{H_0} \right\} < r + \varepsilon$ . Now using the continuity of  $f$ , choose  $0 < \delta < 1$  such that  $0 < t < \delta$  implies  $f(t) < r + \varepsilon_0$ . Let

$$\Delta_{(\mu,u)}^v x \in \chi_{\Delta_{(\mu,u)}^v}^{2I_2} [ \| (d_1(x_1, 0), \dots, d_n(x_n, 0)) \|_p ]^\eta$$

We observe that

$$A(\delta) =$$

$$\left\{ (r, s) \in N \times N : \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f_1 \left\| \left( \left( \left| \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \|_p \right]^{\eta_{mn}} \geq \delta^H \right\}$$

$$\in I_2.$$

Thus if  $(r, s) \notin A(\delta)$  then

$$\frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f_1 \left\| \left( \left( \left| \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \|_p \right]^{\eta_{mn}} < \delta^H$$

$$\Rightarrow \sum_{m=1}^r \sum_{n=1}^s \left[ f_1 \left\| \left( \left( \left| \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \|_p \right]^{\eta_{mn}} < rs \delta^H,$$

$$\Rightarrow \left[ f_1 \left\| \left( \left( \left| \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \|_p \right]^{\eta_{mn}} < \delta^H, \text{ for all } m, n = 1, 2, \dots$$

$$\Rightarrow f_1 \left( \left\| \left( \left( \left| \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right) \|_p \right) < \delta, \text{ for all } m, n = 1, 2, \dots$$

Hence from above inequality and using continuity of  $f$ , we must have



$$f \left( f_1 \left( \left\| \left( \left| \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right) \right) < r + \varepsilon_0, \text{ for all } m, n = 1, 2, \dots$$

$$\sum_{m=1}^r \sum_{n=1}^s \left[ f \left( f_1 \left\| \left( \left| \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right) \right]^{\eta_{mn}} <$$

$$rs \max \left\{ r + \varepsilon_0^H, r + \varepsilon_0^{H_0} \right\} < rs (r + \varepsilon)$$

$$\Rightarrow \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left( f_1 \left\| \left( \left| \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right) \right]^{\eta_{mn}} < r + \varepsilon.$$

Hence we have

$$\left\{ (r, s) \in N \times N : \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left( f_1 \left\| \left( \left| \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right) \right]^{\eta_{mn}} \geq \varepsilon \right\}$$

$\subset A(\delta) \in I_2$ .

(ii) Let  $x \in \chi_{\Delta_{(\mu,u)}^v f_1}^{2I_2} [ \|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p ]^\eta \cap \chi_{\Delta_{(\mu,u)}^v f_2}^{2I_2} [ \|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p ]^\eta$ . Then the fact that

$$\frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ (f_1 + f_2) \left( \left\| \left( \left| \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right) \right]^{\eta_{mn}} \leq$$

$$\frac{D}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f_1 \left( \left\| \left( \left| \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right) \right]^{\eta_{mn}} +$$

$$\frac{D}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f_2 \left( \left\| \left( \left| \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right) \right]^{\eta_{mn}}.$$

This completes the proof.

### 3.7. Theorem

The class of sequence  $\Lambda_{\Delta_{(\mu,u)}^v f}^{2I_2} [ \|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p ]^\eta$  is a sequence algebra

**Proof:** Let  $(\Delta_{(\mu,u)}^v x_{mn}), (\Delta_{(\mu,u)}^v y_{mn}) \in \Lambda_{\Delta_{(\mu,u)}^v f}^{2I_2} [ \|(d_1(x_1, 0), \dots, d_n(x_n, 0))\|_p ]^\eta$  and  $0 < \varepsilon < 1$ . Then the result follows from the following inclusion relation:

$$\left\{ (r, s) \in N \times N : \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left\| \left( \left| \Delta_{(\mu,u)}^v x_{mn} \otimes \Delta_{(\mu,u)}^v y_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right]^{\eta_{mn}} < \varepsilon + K \right\}$$

$$\in I_2$$

$$\supseteq \left\{ \left\{ (r, s) \in N \times N : \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left\| \left( \left| \Delta_{(\mu,u)}^v x_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right]^{\eta_{mn}} < \varepsilon + K \right\} \in I_2 \right\}$$

$$\cap$$

$$\left\{ \left\{ (r, s) \in N \times N : \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \left[ f \left\| \left( \left| \Delta_{(\mu,u)}^v y_{mn} \right| \right)^{1/m+n}, d_1(x_1, 0), \dots, d_n(x_{n-1}, 0) \right\|_p \right]^{\eta_{mn}} < \varepsilon + K \right\} \in I_2 \right\}.$$

Similarly we can prove the result for other cases.

## Conclusion

In numerical analysis, the idea of rough convergence arises very naturally and it has interesting applications on various fields. In this paper, by using the notion of rough  $I$ -convergence, we have established some Orlicz sequence spaces of fuzzy numbers and deliberate some algebraic and topological properties of these spaces.

## References

- [1] Aytar, S., Rough statistical Convergence, *Numerical Functional Analysis Optimization*, 29(3), 291-303, 2008.
- [2] Pal, S.K., Chandra, D., Dutta, S., Rough ideal Convergence, *Hacetatepe Journal Mathematics and Statistics*, 42(6), 633-640, 2013.
- [3] Phu, H.X., Rough convergence in normed linear spaces, *Numerical Functional Analysis Optimization*, 22, 201-224, 2001.
- [4] Kamthan, P.K., Gupta, M., Sequence spaces and series, Lecture notes, Pure and Applied Mathematics, 65 *Marcel Dekker, Inc., New York*, 1981.
- [5] Lindenstrauss, J., Tzafriri, L., On Orlicz sequence spaces, *Israel Journal of Mathematics*, 10, 379-390, 1971.
- [6] Kizmaz, H., On certain sequence spaces, *Canadian Mathematical Bulletin*, 24(2), 169-176, 1981.
- [7] Mursaleen, M., Edely, O.H.H., Statistical convergence of double sequences, *Journal of Mathematical Analysis and Applications*, 288(1), 223-231, 2003.
- [8] B.C. Tripathy, Sen, M., Nath, S.,  $I$ -convergence in probabilistic  $n$ -normed space, *Soft Computing*, 16, 1021-1027, 2012.
- [9] B.C. Tripathy, Hazarika, B., Choudhary, B., Lacunary  $I$ -convergent sequences, *Kyungpook Math. Journal*, 52(4), 473-482, 2012.
- [10] Tripathy, B.C., Sarma, B., Statistically convergent difference double sequence spaces, *Acta Mathematica Sinica*, 24(5), 737-742, 2008.
- [11] Tripathy, B.C., Sarma, B., On  $I$ -convergent Double sequence spaces of fuzzy numbers, *Kyungpook Math. Journal*, 52(2), 189-200, 2012.
- [12] Tripathy, B.C., On statistically convergent double sequences, *Tamkang Journal of Mathematics*, 34(3), 231-237, 2003.
- [13] Tripathy, B.C., Hazarika, B.,  $I$ -convergent sequence spaces associated with multiplier sequence spaces, *Mathematical Inequalities and Applications*, 11(3), 543-548, 2008.