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Approximate Solutions of Nonlinear Volterra Integro-Differential Equations

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Abstract. In this paper, we use sinc-collocation method to obtain approximate solution of a class of nonlinear Volterra integro-differential equations. The introduced method is tested on some nonlinear problems and it seems that the method is a very efficient and powerful tool to obtain approximate solutions of nonlinear integro-differential equations.

Keywords. Sinc methods; Nonlinear Volterra integral equation; Approximate solution.

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1. Introduction

Since sinc methods give a much better rate of convergence and more efficient results than classical polynomial methods in the presence of singularities, they are studied by several authors in the literature. These methods were introduced in [1] and expanded in [2] by Frank Stenger. The sinc functions were first analyzed in [3] and [4]. Sinc-collocation method is used for solving nonlinear Fredholm integro-differential equations in [5]. The authors in [6] use Sinc-Galerkin method and Sinc-Collocation method to solution of nonlinear boundary value problems. In the paper is given by [7], sinc-collocation method develop to obtain numerical solution of linear Fredholm integro-differential equation. In [8], nonlinear boundary value problems are solved by using sinc-Galerkin method. The authors in [9] solve Troesch's problem by using sinc-Galerkin method.
The main purpose of the present paper is to obtain approximate solutions of nonlinear Volterra integro-differential equation

$$y'' + p(x)y' + r(x)y^m = f(x) + \lambda \int_a^x K(x,t)y^s(t)dt \ , \ \ 0 < \alpha < 1 \tag{1}$$

with homogeneous boundary conditions,

$$y(a) = 0 \ , \ y(b) = 0 \tag{2}$$

2. Sinc Functions

In this section, we recall notations and definitions of the sinc function and derive useful formulas that are important for this paper.

Definition 1. The Sinc function is defined on the whole real line by

$$sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & , \quad x \neq 0 \\ 1 & , \quad x = 0. \end{cases} \tag{3}$$

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Definition 2. For $h > 0$ and $k = 0, \mp 1, \mp 2, ...$, the translated sinc function with space node are given by:

$$S(k,h)(x) = sinc\Big(\frac{x-kh}{h}\Big) = \begin{cases} \frac{\sin\left(\pi\frac{x-kh}{h}\right)}{\pi\frac{x-kh}{h}} & , \quad x \neq kh \\ 1 & , \quad x = kh. \end{cases} \tag{4}$$

Definition 3. If $f(x)$ is defined on the real line, then for $h > 0$ the series

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh) sinc\Big(\frac{x-kh}{h}\Big) \tag{5}$$

is called the Whittaker cardinal expansion of f whenever this series converges.

In general, approximations can be constructed for infinite, semi-infinite and finite intervals. To construct approximation on the interval (a,b) the conformal map

$$\phi(z) = \ln\Big(\frac{z-a}{b-z}\Big) \tag{6}$$

is employed. This map carries D_E the eye-shaped domain in the z-plane

$$D_E = \Big\{z = x+iy : |arg\Big(\frac{z-a}{b-z}\Big)| < d \leq \frac{\pi}{2}\Big\} \tag{7}$$

onto the infinite strip D_S

$$D_S \equiv \Big\{w = u+iv : |v| < d \leq \frac{\pi}{2}\Big\} \tag{8}$$

The basis functions on the interval (a,b) are derived from the composite translated sinc functions

$$S_k(z) = S(k,h)(z) \circ \phi(z) = sinc\Big(\frac{\phi(z)-kh}{h}\Big). \tag{9}$$

for $z \in D_E$.

The inverse map of $w = \phi(z)$ is

$$z = \phi^{-1}(w) = \frac{a+be^w}{1+e^w}. \tag{10}$$

The sinc grid points $z_k \in (a,b)$ in D_E will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{a+be^{kh}}{1+e^{kh}}, \qquad k = 0, \pm 1, \pm 2, ... \tag{11}$$

Definition 4. Let D_E be a simply connected domain in the complex plane $\mathbb{C}$, and let ∂D_E denote the boundary of D_E. Let a and b be points on ∂D_E and ϕ be a conformal map D_E onto D_S such that $\phi(a) = -\infty$ and $\phi(b) = \infty$. If the inverse map of ϕ is denoted by φ, define

$$\Gamma = \{\phi^{-1}(u) \in D_E : -\infty < u < \infty\} \tag{12}$$

and $z_k = \varphi(kh),\ k = \mp 1, \mp 2, ...$

Definition 5. Let $B(D_E)$ be the class of functions F that are analytic in D_E and satisfy

$$\int_{\psi(L+u)} |F(z)| dz \to 0, \ \ as \ \ u = \mp\infty, \tag{13}$$

where

$$L = \Big\{iy : |y| < d \leq \frac{\pi}{2}\Big\}, \tag{14}$$

and those on the boundary of D_E satisfy

$$T(F) = \int_{\partial D_E} |F(z)|dz < \infty. \tag{15}$$

Theorem 1. Let Γ be in $(0,1)$ and $F \in B(D_E)$, then for $h > 0$ sufficiently small,

$$\int_\Gamma F(z)dz - h\sum_{j=-\infty}^{\infty} \frac{F(z_j)}{\phi'(z_j)} = \frac{i}{2}\int_{\partial D} \frac{F(z)k(\phi,h)(z)}{sin(\pi\phi(z)/h)}dz \equiv I_F \tag{16}$$

where

$$|k(\phi,h)|_{z\in\partial D} = \left|e^{\left[\frac{i\pi\phi(z)}{h} sgn(Im\phi(z))\right]}\right|_{z\in\partial D} = e^{\frac{-\pi d}{h}} \tag{17}$$

For the term of fractional in eq.(1), the infinite quadrature rule must be truncated to a finite sum. The following theorem indicates the conditions under which an exponential convergence results.

Theorem 2. If there exist positive constants α, β and C such that

$$\left|\frac{F(x)}{\phi'(x)}\right| \le C \left\{ \begin{array}{lll} e^{-\alpha|\phi(x)|} & , & x \in \psi((-\infty,\infty)) \\ e^{-\beta|\phi(x)|} & , & x \in \psi((0,\infty)). \end{array} \right. \tag{18}$$

then the error limit for the quadrature rule (16) is

$$\left|\int_\Gamma F(x)dx - h\sum_{j=-M}^{N} \frac{F(x_j)}{\phi'(x_j)}\right| \le C\left(\frac{e^{-\alpha Mh}}{\alpha} + \frac{e^{-\beta Nh}}{\beta}\right) + |I_F| \tag{19}$$

The infinite sum in (16) is cut with the help of to achieve (18). With the help of the choices

$$h = \sqrt{\frac{\pi d}{\alpha M}}\ and\ N \equiv \left\lfloor \frac{\alpha M}{\beta} + 1 \right\rfloor \tag{20}$$

knowing that $\lfloor . \rfloor$ is the integer part of the expression and M is the integer value that determines the size of the grid,

$$\int_\Gamma F(x)dx = h\sum_{j=-M}^{N} \frac{F(x_j)}{\phi'(x_j)} + O\left(e^{-(\pi\alpha d M)^{1/2}}\right). \tag{21}$$

Lemma 1. If ϕ is the conformal 1-1 mapping which simply connects the domain D_E onto D_S. Then

$$\begin{aligned} \delta_{jk}^{(0)} &= [S(j,h)o\phi(x)]|_{x=x_k} \left\{ \begin{array}{lll} 1 & , & j=k \\ 0 & , & j\neq k. \end{array} \right. \\ \delta_{jk}^{(1)} &= h\frac{d}{d\phi}[S(j,h)o\phi(x)]\Big|_{x=x_k} \left\{ \begin{array}{lll} 0 & , & j=k \\ \frac{(-1)^{k-j}}{k-j} & , & j\neq k. \end{array} \right. \\ \delta_{jk}^{(2)} &= h^2\frac{d^2}{d\phi^2}[S(j,h)o\phi(x)]\Big|_{x=x_k} \left\{ \begin{array}{lll} -\frac{\pi^2}{3} & , & j=k \\ \frac{-2(-1)^{k-j}}{(k-j)^2} & , & j\neq k. \end{array} \right. \end{aligned} \tag{22}$$

3. The Sinc-Collocation Method

We assume an approximate solution for $y(x)$ in problem (1) and (2) by the finite expansion of sinc basis functions

$$y_n(x) = \sum_{k=-M}^{N} c_k S_k(x), \qquad n = M+N+1 \tag{23}$$

where $S_k(x)$ is the the function $S(k,h)o\phi(x)$. The unknown coefficients c_k in (23) are determined by sinc-collocation method. For this purpose, the first and the second derivatives of $y_n(x)$ are given by

$$\frac{d}{dx}y_n(x) = \sum_{k=-M}^{N} c_k \phi^{'}(x)\frac{d}{d\phi}S_k(x) \tag{24}$$

$$\frac{d^2}{dx^2}y_n(x) = \sum_{k=-M}^{N} c_k\Big(\phi^{''}(x)\frac{d}{d\phi}S_k(x)+(\phi^{'})^2\frac{d^2}{d\phi^2}S_k(x)\Big) \tag{25}$$

Application of (21) to the kernel integral in (1) gives the following lemma.

Lemma 2. The following relation holds

$$\int_a^{x_j} K(x,t)y^s(t)dt \cong h\sum_{k=-M}^{N}\delta_{jk}^{(-1)}\frac{K(x_j,t_k)}{\phi'(t_k)}y_k^s \tag{26}$$

where

$$\delta_{jk} = \int_0^{j-k}\frac{sin\pi t}{\pi t}dt$$

$$\delta_{jk}^{(-1)} = \frac{1}{2}+\delta_{jk} \tag{27}$$

and y_k denotes an approximate value of $y(t_k)$.
Replacing each term of (1) with the approximation given in (23)-(27), multiplying the resulting equation by $\{1/(\phi^{'})^2\}$ and setting $x = x_j$, we obtain the following nonlinear system

$$\sum_{k=-M}^{N} c_k\Big\{\frac{d^2}{d\phi^2}S_k+\Big[p\Big(\frac{1}{\phi'}\Big)-\Big(\frac{1}{\phi'}\Big)^{'}\Big](x_j)+c_j^m\Big(r\Big(\frac{1}{\phi'}\Big)^2\Big)(x_j)-\lambda h\sum_{k=-M}^{N}\delta_{jk}^{(-1)}c_k^s\frac{K(x_j,t_k)}{\phi'(t_k)}\Big(\frac{1}{\phi'(x_j)}\Big)^2 = \Big(f\Big(\frac{1}{\phi'}\Big)^2\Big)(x_j) \tag{28}$$

where $j = -M,...,N$. By using Lemma1, we know that

$$\delta_{jk}^{(0)} = \delta_{kj}^{(0)}\ ,\ \delta_{jk}^{(1)} = -\delta_{kj}^{(1)}\ ,\ \delta_{jk}^{(2)} = \delta_{kj}^{(2)} \tag{29}$$

then we obtain the following theorem.

Theorem 3. If the assumed approximate solution of boundary value problem (1) is (23), then the discrete sinc-collocation system for the determination of the unknown coefficients $\{c_k\}_{k=-M}^{N}$ is given by

$$\sum_{k=-M}^{N} c_k\Big\{\frac{1}{h^2}\delta_{jk}^{(2)}+\frac{1}{h}\Big[\Big(\frac{1}{\phi'}\Big)^{'}-p\Big(\frac{1}{\phi'}\Big)\Big](x_j)\delta_{jk}^{(1)}+c_j^m\Big(r\Big(\frac{1}{\phi'}\Big)^2\Big)(x_j)-\lambda h\sum_{k=-M}^{N}\delta_{jk}^{(-1)}c_k^s\frac{K(x_j,t_k)}{\phi'(t_k)}\Big(\frac{1}{\phi'(x_j)}\Big)^2 = \Big(f\Big(\frac{1}{\phi'}\Big)^2\Big)(x_j) \tag{30}$$

where $j = -M,...,N$.

Now we define some notations to represent in the matrix-vector form for system (30). Let $D(y)$ denotes a diagonal matrix whose diagonal elements are $y(x_{-M}), y(x_{-M+1}),,...,y(x_N)$ and non-diagonal elements are zero, and

$$E = \frac{K(x_j,t_k)}{\Big(\phi'(x_j)\Big)^2\phi'(t_k)} \tag{31}$$

denote a matrix and also let $I^{(i)}$ denotes the matrices

$$I^{(i)} = \Big[\delta_{jk}^{(i)}\Big]\quad,\quad i=-1,0,1,2 \tag{32}$$

where $D, E, I^{(-1)}, I^{(0)}, I^{(1)}$ and $I^{(2)}$ are square matrices of order nXn. In order to calculate unknown coefficients c_k in nonlinear system (30), we rewrite this system by using the above notations in matrix-vector form as

$$A_1 C + A_2 C^m + A_3 C^s = B \tag{33}$$

where

$$\begin{aligned}
A_1 &= \frac{1}{h^2} I^{(2)} + \frac{1}{h} D\Big(\Big(\frac{1}{\phi'}\Big)' - p\Big(\frac{1}{\phi'}\Big)\Big) I^{(1)} \\
A_2 &= D\Big(r\Big(\frac{1}{\phi'}\Big)^2\Big) I_{(0)} \\
A_3 &= -h\lambda (E \circ I^{(-1)}) \\
B &= \left(\Big(f\Big(\frac{1}{\phi'}\Big)^2(x_{-M}), \Big(f\Big(\frac{1}{\phi'}\Big)^2(x_{-M+1}), \ldots, \Big(f\Big(\frac{1}{\phi'}\Big)^2\Big)(x_N)\right)^T \\
C &= (c_{-M}, c_{-M+1}, \ldots, c_N)^T \\
C^m &= (c^m_{-M}, c^m_{-M+1}, \ldots, c^m_N)^T \\
C^s &= (c^s_{-M}, c^s_{-M+1}, \ldots, c^s_N)^T
\end{aligned} \tag{34}$$

The notation "$\circ$" denotes the Hadamard matrix multiplication. Now we have nonlinear system of n equations in the n unknown coefficients given by (33). When it is solved by Newton's method, we can obtain the unknown coefficients c_k that are necessary for approximate solution in (23).

4. Computational Examples

In this section, two problems that have homogeneous boundary conditions will be tested by using the present method via Mathematica10 on a personal computer. In all the examples, we take $d = \pi/2, \alpha = \beta = 1/2, N = M$.

Example 1. Consider nonlinear Volterra integro-differential equation in the following form

$$y''(x) + y^2(x) = f(x) - 2\int_0^x K(x,t) y^3(t)dt \tag{35}$$

subject to the homogeneous boundary conditions $y(0) = 0,\ y(1) = 0$ where $f(x) = -\frac{1}{55}x^{11} + \frac{1}{12}x^9 - \frac{1}{7}x^7 + x^6 + \frac{1}{10}x^5 - 2x^4 + x^2 - 6x$ and $K(x,t) = x - t$. The exact solution of this problem is $y(x) = x(1 - x2)$. The numerical solutions which are obtained by using the present method for this problem are presented in Table 1 . Additionally, the graphics of the exact and approximate solutions for different values of M are given in Figure 1.

Example 2. Consider nonlinear Volterra integro-differential equation in the following form

$$y''(x) + \frac{1}{x-1} y^3(x) = f(x) + \int_0^x K(x,t) y^2(t)dt \tag{36}$$

subject to the homogeneous boundary conditions $y(0) = 0,\ y(1) = 0$ where $f(x) = x^{11} - \frac{181}{90}x^{10} + \frac{37}{36}x^9 - \frac{1}{56}x^8 + 12x^2 + 6x$ and $K(x,t) = x - t$. The exact solution of this problem is $y(x) = x^3(x-1)$. The numerical solutions which are obtained by using the present method for this problem are presented in Table 2. Additionally, the graphics of the exact and approximate solutions for different values of M are given in Figure 2.

TABLE 1. Numerical results for $M = 5$ and $M = 50$

x	Exact Solution	Error
0	0	0
0.1	0.099	5.86 × e-3
0.2	0.192	2.38 × e-3
0.3	0.273	2.96 × e-3
0.4	0.336	3.17 × e-3
0.5	0.375	1.11 × e-4
0.6	0.384	3.04 × e-3
0.7	0.357	4.27 × e-3
0.8	0.288	3.68 × e-3
0.9	0.171	2.49 × e-3
1.0	2.2 e-2	0

x	Exact Solution	Error
0	0	0
0.1	0.099	1.68 e-9
0.2	0.192	7.89 e-9
0.3	0.273	8.90 e-9
0.4	0.336	6.62 e-9
0.5	0.375	8.97 e-9
0.6	0.384	1.12 e-9
0.7	0.357	5.89 e-9
0.8	0.288	4.28 e-9
0.9	0.171	3.98 e-9
1.0	2.2 e-2	0

TABLE 2. Numerical results for $M = 5$ and $M = 50$

x	Exact Solution	Error
0	0	0
0.1	-0.0009	3.00 × e-4
0.2	-0.0064	1.30 × e-3
0.3	-0.0189	2.17 × e-3
0.4	-0.0384	1.00 × e-4
0.5	-0.0625	2.14 × e-3
0.6	-0.0864	5.20 × e-3
0.7	-0.1029	5.05 × e-3
0.8	-0.1024	1.47 × e-5
0.9	-0.0729	3.67 × e-3
1.0	0	0

x	Exact Solution	Error
0	0	0
0.1	-0.0009	7.67 × e-10
0.2	-0.0064	3.57 × e-9
0.3	-0.0189	1.29 × e-10
0.4	-0.0384	6.41 × e-9
0.5	-0.0625	1.80 × e-9
0.6	-0.0864	3.26 × e-9
0.7	-0.1029	5.57 × e-9
0.8	-0.1024	9.42 × e-9
0.9	-0.0729	1.65 × e-10
1.0	0	0

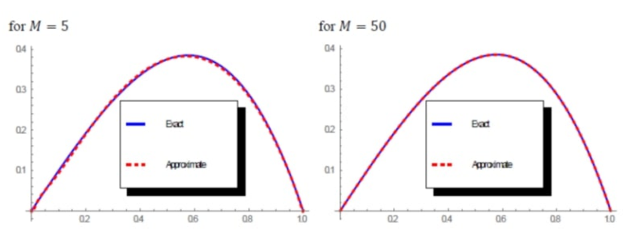

FIGURE 1. The graphics of the exact and approximate solutions for different values of M

Conclusion

In this paper, sinc-collocation method is used to obtain approximate solution of a general nonlinear integro-differential equation with boundary conditions. In order to illustrate the accuracy and effective of the

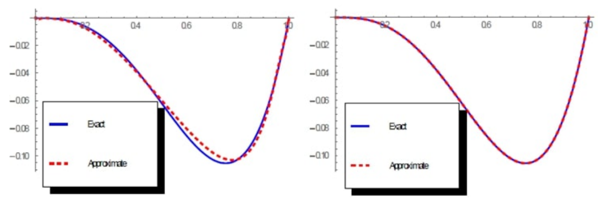

FIGURE 2. The graphics of the exact and approximate solutions for different values of M

method, it is applied to some examples and obtained results are compared with the exact ones. The comparisons in table and graphical forms show that the approximate solutions converge the exact ones when it is increased that the number of sinc grid points N and the present method is a powerful tool for solving nonlinear integro-differential equations with boundary conditions.

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