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The Similarity Orbits in $\mathbf{R}^3$

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Abstract. Let $(G,*)$ be a group and X be a nonempty set and the group action $G:X$ be given. In this paper we studied the $G-$orbits, the invariant subspaces, in R^3 regarding as $G=S(3)$, all similarity transformations' group in three dimensional Euclidean space, and all subgroups of it.

Keywords. G-orbits; Invariant subsets; Similarity groups.

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1. Introduction

Let $(G,*)$ be a group and X be a given nonempty set. For any element x, the set of images $g*x$ under the all $g\in G$ is called $G-$orbit of the element x. The smallest $G-$invariant subset containing element x is $G-$orbit of x. The $G-$invariant subset also covers the $G-$orbit Gx if covers element x.The problem of finding invariant subspaces is widely used in mathematics. Especially it is a concept used in operator theory in Hilbert spaces and used for solutions of nonlinear differential equations and for complete continuous operators. [[**?**],[**?**],[**?**], [**?**]].

The first group action studied was the action of the Galois group on the roots of a polynomial. However, there are a lot of examples and applications of group actions in many areas of mathematics as well as chemistry and physics. In general, an orbit may be of any dimension, up to the dimension of the Lie group. If the Lie group G is compact, then its orbits are submanifolds. [[**?**]] .

It is seen that orbit spaces has been studied in many areas in invariant theory. [[**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**]]. In 1897 E. Study gave the full invariant system of the points for the group $O(n)$. Then Hermann Weyl developed this subject later him [[**?**]]. In 1988, Djavvat Khadjiev, R. Aripov solved this problem for the whole Euclidean movement group$E(n)$ [[**?**]]. Then Weyl representations have been applied to a wide variety of areas. These applications have been expanded in many fields ranging from the classification of bacteria to the problems of perception. For example, E. Cassier [[**?**]] gave the invariants of the transformation group as perceptual characterizations. Later Hoffman [[**?**], [**?**]] developed the theory of perception based on Lie transformation groups and Lie algebras. They were followed by Chan & Chan [[**?**], [**?**]] and Leyton [[**?**]]. Other studies on invariant parameterization and curves theory can be demonstrated [[**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**]]. In this paper we studied the $G-$orbits, the invariant subspaces, in R^3 regarding as $G=S(3)$ and all subgroups of it.

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2. The Group Actions

Definition 1. Let $(G,*)$ be a group and X be a nonempty set. Let the transformation $\varphi : G\times X \to X$ be given. If following conditions

i) $\varphi(g_1,\ \varphi(g_2,x)) = \varphi(g_1 * g_2,x), \quad \forall g_1,g_2\in G$ and $\forall x\in X$

ii) $\varphi(e,\ x) = x, \quad for \forall x\in X$ where $e\in G$ is identity element.

satisfies then the transformation φ is called Group action of the group G on X. This action is denoted by $G:X$ and the image $\varphi(g,\ x)$ is stated as gx briefly.

Example 2.1. Let G be choosen as the linear homotethies' sets in three dimensional Euclidean spaces as follows

$$G = LH(3) = \left\{ \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} : \lambda \in R^+ \right\}$$

This set is a group with binary operation of the product of the matrices. The action of this group on E^3. Let $G:E^3$ be defined by $\varphi : G\times E^3 \to E^3$ such that $(g,x) \overset{\varphi}{\to} \varphi(g,\ x) = gx$ where $g = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. It is easily shown that the action of $\varphi : G\times E^3 \to E^3$ is really the group action:

For $g_1 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$, $g_2 = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$, and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in E^3$

$$\begin{aligned}
\text{i) } \varphi(g_1,\ \varphi(g_2,x)) &= \varphi\left(\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}, \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) \\
&= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} . \begin{bmatrix} \lambda_2 x_1 \\ \lambda_2 x_2 \\ \lambda_2 x_3 \end{bmatrix} = \begin{bmatrix} \lambda_1\lambda_2 x_1 \\ \lambda_1\lambda_2 x_2 \\ \lambda_1\lambda_2 x_3 \end{bmatrix} \\
&= \varphi\left(\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} . \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) \\
&= \varphi(g_1 g_2,\ x)
\end{aligned}$$

$$\text{ii) } \varphi(e,\ x) = \varphi\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x$$

for every $x\in E^3$.

When any $g\in G$ be choosen and be fixed the transformation $\varphi : G\times X \to X$; $(g,x) \overset{\varphi}{\to} \varphi(g,\ x) = gx$ is reduced to the transformation $\varphi(g,\ .) : X \to X$; $x \overset{\varphi(g,\ .)}{\to} \varphi(g,\ x) = gx$.

Definition 2. Let G be a group and the group actions of G on the sets X_1 and X_2 such that $\varphi_1 = G : X_1$ and $\varphi_2 = G : X_2$ be given. If there exist a one-to-one and onto transformation $F : X_1 \longrightarrow X_2$ such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{F} & X_2 \\ \varphi_1(g,\ .)\downarrow & & \downarrow\varphi_2(g,\ .) \\ X_1 & \xrightarrow{F} & X_2 \end{array}$$

is commutative or such that $F \circ \varphi_1(g,\ .) = \varphi_2(g,\ .) \circ F$ is satisfy, then the actions $\varphi_1 = G : X_1$ and $\varphi_2 = G : X_2$ are called equivalent group actions.

Example 2.2. Let $G = LH(3)$ be choosen and $X_1 = \{(x,y,z) : z > 0\} \subset R^3$ and $X_2 = \{(x,y,z) : z < 0\} \subset R^3$ be given. Two group actions of the group G are

$\varphi_1 = G : X_1$ such that

$$\varphi_1(g,\ (x,y,z)) = \varphi_1\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix},\ (x,y,z)\right)$$

$$= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}$$

Example 2.3. and $\varphi_2 = G : X_2$ such that

$$\varphi_2(g,\ (x,y,z)) = \varphi_2\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix},\ (x,y,z)\right)$$

$$= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}$$

be considered. In this case there exist a one-to-one and onto transformation $F : X_1 \longrightarrow X_2$ such that $F(x,y,z) = (x,y,-z)$ can be considered. Hence the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{F} & X_2 \\ \varphi_1(g,\ .)\downarrow & & \downarrow\varphi_2(g,\ .) \\ X_1 & \xrightarrow{F} & X_2 \end{array}$$

is commutative. So these group actions are equivalent.

3. The $G-$orbits

Definition 3. Let G be a group and the group action $\varphi = G : X$ and the subset $H \subset X$ be given. the subset H is called "$G-$**invariant subset**" if $\varphi(g,h) \in H$ for every $h \in H$ and for every $g \in G$.

Considering $H = \{x_0\} \subset X$ in this definition, i.e. the point x_0 is called "*$G-$invariant point*" if $gx_0 = x_0$ for every $g \in G$.

Definition 4. Let G be a group and the group action $\varphi = G : X$ be given. For any element $x \in X$ the set $Gx = \{gx : g \in G\}$ is called $G-$orbit of the element x.

Proposition 1. Let G be a group and the group action $\varphi = G : X$ be given. The $G-$orbit of the element x is *$G-$invariant subset of X.*

Proof. To prove that $Gx \subset X$ is $G-$invariant subset, we must shown $gy \in Gx$ for every $y \in Gx$ and for every $g \in G$. Let $y \in Gx$ be given. ın this case there exist $g_1 \in G$ such that $y = g_1 x$ satisfies. So $gy = g(g_1 x)$ can be written. From defining group action it is true that $g(g_1 x) = (gg_1)x$. From that G is a group , it can be seen easily $gy = g^* x$ by writing g^* instead of gg_1. ■

Proposition 2. Let G be a group and the group action $\varphi = G : X$ be given. There is no any $G-$invariant subset of Gx different from itself.

Proof. Let $H \subset Gx$ be a $G-$invariant subset of Gx. we must prove that $H = Gx$. Since $e \in G$ is identity element, $x \in Gx$. Let $y \in H$ be given. In this case there exist $g_1 \in G$ such that $y = g_1 x \in H$ is satisfies. it is seen that $gy = g(g_1 x) \in H$ for every $g \in G$, since H is a *$G-$invariant subset. So iti is also true for* $g = g_1^{-1}$. *it means* $gy = g_1^{-1}(g_1 x) = x \in H$ is obtained. As a result of this $Gx \subset H$ is satisfies since $x \in H$ is obtained when $x \in Gx$. So $H = Gx$ satisfies. ■

Corollary 1. Let G be a group and the group action $\varphi = G : X$ be given. Then $Gx = Ga$ for any $a \in Gx$.

Proof. it can be easily peoved. ■

Proposition 3. Let G be a group and the group action $\varphi = G : X$ be given. Then $Gx = Gy$ for different elements x and y of X if $Gx \cap Gy \neq \varnothing$. with other statement, $Gx \cap Gy \neq \varnothing$ if $Gx \neq Gy$ for different elements x and y of X.

Proof. Let $Gx \cap Gy \neq \varnothing$ be supposed. in this case there exist $a \in Gx$ and $a \in Gy$. From previous corollary $Ga = Gx$ and $Ga = Gy$ is concluded . So $Gx = Gy$ is obtained. ■

Definition 5. Let G be a group and the group action $\varphi = G : X$ be given. For $x, y \in X$, if there exist $g \in G$ such that $y = gx$ satisties, then the points x and y are called $G-$ equivalent points and denoted by $x \overset{G}{\sim} y$.

Proposition 4. the definition of $G-$equivalence of points is an equivalence relationship.

As a result of the definition of $G-$equivalence of points, the equivalence class of an element x according to the relationship $\overset{G}{\sim}$ are $G-$orbit Gx.

4. The Similarity Group $G = S(3)$ and its all subgroups in R^3

1- **Ortogonal Transformations'Group** $O(3)$

This group is a group of all rotations and reflections. Ortogonal Transformations' group is the same of the linear isometries' group. For any $g \in O(3)$, the determinant of g is equal ± 1.The rotations are depend on the angles between rotating axisses $\{e_1^*, e_2^*, e_3^*\}$ and fixed axisses $\{e_1, e_2, e_3\}$ the of the frames $\{e_1, e_2, e_3\}$. The angle θ_{ij}, is an angle between the axisses $\{e_i^*, e_j\}$ So this group can be stated as

$$O(3)=\left\{f:R^3\longrightarrow R^3: f(x)=gx;\;\; g^T=g^{-1}\right\}$$
$$=\left\{\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}.\begin{bmatrix}\cos\theta_{11}&\cos\theta_{21}&\cos\theta_{31}\\\cos\theta_{12}&\cos\theta_{22}&\cos\theta_{32}\\\cos\theta_{13}&\cos\theta_{23}&\cos\theta_{33}\end{bmatrix}:\theta_{ij}\in R\right\}$$

2- **Special Ortogonal Transformations' Group** $SO(3)$

This group is a group of all rotations. For any $g\in SO(3)$, the determinant of g is equal $+1$.The rotations are depend on the angles between rotating axisses $\left\{e_1^*,e_2^*,e_3^*\right\}$ and fixed axisses $\{e_1,e_2,e_3\}$ the of the frames $\{e_1,e_2,e_3\}$. The angle θ_{ij}, is an angle between the axisses $\left\{e_i^*,e_j\right\}$ So this group can be stated as

$$SO(3)=\left\{f:R^3\longrightarrow R^3: f(x)=gx;\;\; g^T=g^{-1};\det g=+1\right\}$$
$$=\left\{\begin{bmatrix}\cos\theta_{11}&\cos\theta_{21}&\cos\theta_{31}\\\cos\theta_{12}&\cos\theta_{22}&\cos\theta_{32}\\\cos\theta_{13}&\cos\theta_{23}&\cos\theta_{33}\end{bmatrix}:\theta_{ij}\in R\right\}$$

3- **Translations'Group** $Tr(3)$

This group is a group of all translations.

$$Tr(3)=\left\{f:R^3\longrightarrow R^3: f(x)=x+b;\;\; b\in R^3\right\}$$

4- **Euclid Transformations' Group** $E(3)$

This group is a group of all translations, rotations, rotations and translations, reflections, reflections and translations. this group is the same of all isometries' group. Euclid transformations can be stated as a compositition of a translation and an orthogonal transformation. So this group can be stated as

$$E(3)=\left\{f:R^3\longrightarrow R^3: f(x)=gx+b;\; g\in O(3);\;\; b\in R^3\right\}$$

5- **Special Euclid Tranformations' Group** $SE(3)$

This group is a group of all translations, rotations, rotations and translations. Special Euclid transformations can be stated as a compositition of a translation and a special orthogonal transformation. So this group can be stated as

$$SE(3)=\left\{f:R^3\longrightarrow R^3: f(x)=gx+b;\; g\in SO(3);\;\; b\in R^3\right\}$$

6- **Linear Homotethies' Group** $LH(3)$

This group is a group of all central dilations or radyal transformations. Linear homotethies are the transformations that its center of homotethy are origin. So this group can be stated as

$$LH(3)=\left\{f:R^3\longrightarrow R^3: f(x)=\lambda x;\; \lambda\in R^+\right\}$$

7- **Homotethies' Group** $H(3)$

The hometethy transformation can be defined as $f(x)=a+\lambda(x-a)$. where a is called homotethy center of f . This transformation can be stated as $f(x)=\lambda x+b$ for $\lambda\neq 1$ where the center of homnotethy is $a=\frac{b}{1-\lambda}$ So this group can be stataed as

$$H(3)=\left\{f:R^3\longrightarrow R^3: f(x)=\lambda x+b;\; \lambda\in R^+;b\in R^3\right\}$$

7- **Homotethies' Group** $H(3)$

The hometethy transformation can be defined as $f(x)=a+\lambda(x-a)$. where a is called homotethy center of f . This transformation can be stated as $f(x)=\lambda x+b$ for $\lambda\neq 1$ where the center of homnotethy is $a=\frac{b}{1-\lambda}$ So this group can be stataed as

$H(3)=\left\{f:R^3\longrightarrow R^3: f(x)=\lambda x+b;\ \lambda\in R^+; b\in R^3\right\}$

8- **Linear Similarities' Group** $LS(3)$

Linear similarity transformations can be stated as a compositition of a linear homotethy transformation and an orthogonal transformation. So linear similarity transformations can be stated as

$LS(3)=\left\{f:R^3\longrightarrow R^3: f(x)=\lambda gx;\ \lambda\in R^+;\ \ g\in O(3)\right\}$

9-**Special Linear Similarities' Group** $SLS(3)$

Special linear similarity transformations can be stated as a compositition of a linear homotethy transformation and a special orthogonal transformation. So linear similarity transformations can be stated as

$SLS(3)=\left\{f:R^3\longrightarrow R^3: f(x)=\lambda gx;\ \lambda\in R^+;\ \ g\in SO(3)\right\}$

10- **Special Similarities' Group** $SS(3)$

Specialr similarity transformations can be stated as a compositition of a special linear similarity transformation and a translation. In other words a special similarity transformation can be stated as a compositition of a linear homotethy transformation, a special orthogonal transformation, and a translation. So special similarity transformations can be stated as

$SS(3)=\left\{f:R^3\longrightarrow R^3: f(x)=\lambda gx+b;\ \lambda\in R^+;\ \ g\in SO(3);\ \ b\in R^3\right\}$

11- **All Similarities' Group** $S(3)$

Similarity transformations can be stated as a compositition of a linear similarity transformation and a translation. In other words a similarity transformation can be stated as a compositition of a linear homotethy transformation, an orthogonal transformation, and a translation. So similarity transformations can be stated as

$S(3)=\left\{f:R^3\longrightarrow R^3: f(x)=\lambda gx+b;\ \lambda\in R^+;\ \ g\in O(3);\ \ b\in R^3\right\}$

5. The $G-$orbits for $G=S(3)$ and its all subgroups

Now let us obtain the $G-$orbits of R^3 regarding as $G=S(3)$ and its all subgroups.

Theorem 1. The $O(3)-$orbits of R^3 consist of origin, and the surfaces of sphere with radius r. $(r\in R^+)$

Proof. For $x=0$, the $G-$orbit is $G0=\{g.0: g\in O(3)\}=\{0\}$

Let $x=(x_1,x_2,x_3)$ be a nonzero vector of R^3. Then, the $G-$orbit is $Gx=\{gx: g\in O(3)\}$. For the orthogonal transformations f, it is true that $\|f(x)\|=\|x\|$. So $\|gx\|=\|x\|$ satisfies. Then Choosing $\|x\|=r$, the $G-$orbit of x is the set consisting of the vectors x such that $x_1^2+x_2^2+x_3^2=r^2$.(see. Fig.1) ■

Theorem 2. The $SO(3)-$orbits of R^3 consist of origin, and the surfaces of sphere with radius r. $(r\in R^+)$

Proof. It can be easily seen similarly previous theorem. (see. Fig.1) ■

Theorem 3. The $Tr(3)-$orbit of R^3 is space R^3 itself.

Proof. Since any element x can be translated to any element of the space R^3 the $Tr(3)-$orbit of R^3 is space R^3 itself. ■

Theorem 4. The $E(3)-$orbit of R^3 is space R^3 itself.

Proof. Similarly Since any element x can be rotated and reflected and then translated to any element of the space R^3 the $E(3)-$orbit of R^3 is space R^3 itself. ■

Theorem 5. The $SE(3)-$orbit of R^3 is space R^3 itself.

Proof. Similarly Since any element x can be rotated and then translated to any element of the space R^3 the $SE(3)-$orbit of R^3 is space R^3 itself. ■

Theorem 6. The $LH(3)-$orbits of R^3 consist of origin, and rays distributed from origin.

Proof. If $x=0$ then the $G-$orbit of 0 is $G0=\{g.0 : g\in O(3)\}=\{0\}$

If $x\neq 0$ then the $G-$orbit of x is $Gx=\{\lambda.x : \lambda>0\}$. So $LH(3)-$orbits of R^3 consist of origin, and rays distributed from origin. (See Fig. 2) ■

Theorem 7. The $H(3)-$orbit of R^3 is space R^3 itself.

Proof. Since any element x can be dilated and then translated to any element of the space R^3 the $H(3)-$orbit of R^3 is space R^3 itself. ■

Theorem 8. The $LS(3)-$orbits of R^3 consist of two orbits: origin, other part of the space R^3 except of the origin.

Proof. If $x=0$ then the $G-$orbit of 0 is $G0=\{\lambda.g.0 : \lambda>0, g\in O(3)\}=\{0\}$

If $x\neq 0$ then the $G-$orbit of x is $Gx=\{\lambda.g.x : \lambda>0, g\in O(3)\}=R^3\setminus\{0\}$. So $LS(3)-$orbits of R^3 consist of origin, and other part of the space R^3 except of the origin. ■

Theorem 9. The $SLS(3)-$orbits of R^3 also consist of two orbits: origin, other part of the space R^3 except of the origin.

Proof. If $x=0$ then the $G-$orbit of 0 is $G0=\{\lambda.g.0 : \lambda>0, g\in SO(3)\}=\{0\}$

If $x\neq 0$ then the $G-$orbit of x is $Gx=\{\lambda.g.x : \lambda>0, g\in SO(3)\}=R^3\setminus\{0\}$. So $SLS(3)-$orbits of R^3 consist of origin, and other part of the space R^3 except of the origin. ■

Theorem 10. The $SS(3)-$orbit of R^3 is space R^3 itself.

Proof. Similarly Since any element x can be rotated and dilated and then translated to any element of the space R^3 the $E(3)-$orbit of R^3 is space R^3 itself. ■

Theorem 11. The $S(3)-$orbit of R^3 is space R^3 itself.

Proof. Similarly Since any element x can be rotated or reflected and dilated and then translated to any element of the space R^3 the $S(3)-$orbit of R^3 is space R^3 itself. ■

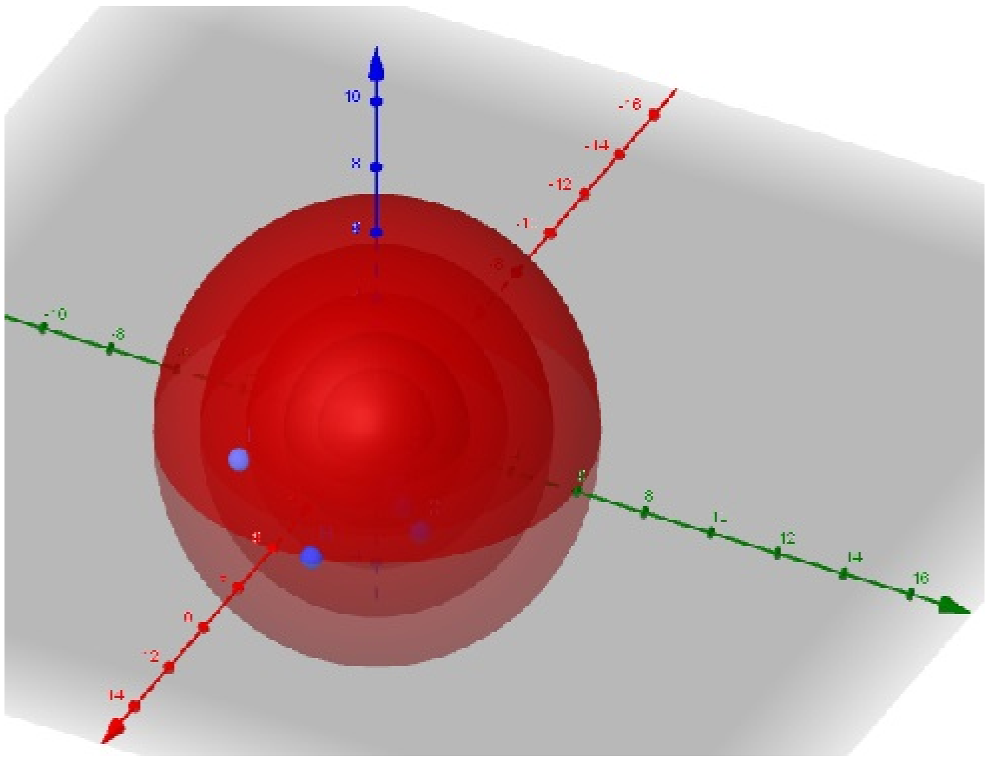

Fig. 1: The $O(3)-$orbits and the $SO(3)-$orbits of R^3

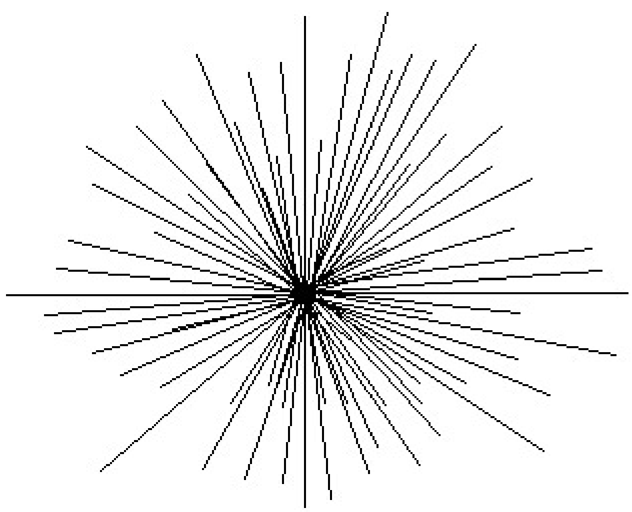

Fig. 2: The $LH(3)-$orbits of R^3

References

[1] Rowland, T., "Group Action." From MathWorld–A Wolfram Web Resource, created by Eric W. Weisstein. http://mathworld.wolfram.com/GroupAction.html

[2] Kolmogoroff, A. N., "Interpolation und Extrapolation", *Bull. Acad. Sci. U.S.S.R. Ser. Math.*, 3-14, 1941.

[3] Wiener, N., Extrapolation, Interpolation and Smoothing of Stationary Time Series with Engineering Applications, *MIT Press*, New York, 1949

[4] Aronszajn, N., Stmith, K. T., "Invariant Subspaces of completely continuous operators", *Annals of Mathematics*, 60(2), 1954.

[5] Galaktionov, V. A., "Invariant subspaces and new explicit solutions to evolution equations with quadratic nonlinearities", *Proceedings of the Royal Society of Edinburgh*, 225-246,1995.

[6] Weyl, H., The Classical Groups, Their Invariants and Representations, 2nd ed., Princeton University Press, *Princeton*, 1946.

[7] Khadjiev, D., An Application of the Invariant Theory to the Differential Geometry of Curves(in Russian), *Tashkent*, 1988.

[8] Cassier, E. T., "The concept of group and the theory of perception", *Philosophy and Phenomenological Research*, 5, 1-35, 1944.

[9] Hoffman, W. C., "The Lie algebra of visual perception", *Journal of Mathematical Psychology*, 3, 65-98, 1966.

[10] Hoffman, W. C., The Lie transformation group approach to visual neuropsychology, Formal theories of visual perception, Chichester, *Wiley*, 1978.

[11] Chan M., Chen, K., "A transformational analysis of form recognition under plane isometries", *Journal of Mathematical Psychology*, 26(3), 237-251, 1982.

[12] Chan M., Chen, K., "A mental space similarity Group model of Shape constancy", *Journal of Mathematical Psychology*, 43, 410-432, 1999.

[13] Leyton, M. A., "Theory of information structure: A theory of perceptual organization", *Journal of Mathematical Psychology*, 30, 257-305, 1986.

[14] Asil, V., Korpinar, T., Bas, S., "New Parametric Representation of a Surfaces B-Pencil with a Common Line of Curvature", *Siauliai Math. Semin.*, 9(17), 5-14, 2014.

[15] Korpinar, T., Bas, S., "Characterization of Quaternionic Curves by Inextensible Flows", *Prespactime Journal*, 7(12), 1680-1684, 2016.

[16] Oren, I., Khadjiev, D., Peksen, O., "Global invariants of paths and curves with respect to similarity groups in the two-dimensional Euclidean space and their applications to mechanics", *International Journal of Geometric methods in modern Physics*, 15, 1-32, 2018.

[17] Oren, I., Khadjiev, D., Peksen, O., "Generating systems of differential invariants and the theorem on existence for curves in the pseudo-Euclidean geometry", *Turkish Journal of Mathematics*, 37(1), 80-94, 2013.

[18] Sagiroglu, Y., "Equi-affine differential invariants of a pair of curves", *TWMS Journal of Pure and Applied Mathematics*, 6, 238-245, 2015.

[19] Yapar, Z., Sagiroglu, Y., "Curvature Motion On Dual Hyperbolic Unit Sphere H_0^2", *Journal of Applied Mathematics and Physics*, 2(8), 828-826, 2014.

[20] Sartori, G., "A theorem on orbit structures (strata) of compact linear Lie groups", *Journal of Mathematical Physics* 24, 1983.

[21] Symonds, P., "The orbit space of the p-subgroup complex is contractible", *Commentarii Mathematici Helvetici*, 73, 400–405, 1998.

[22] Odani, K., "Generic homeomorphisms have the pseudo-orbit tracing property", *Proc. Amer. Math. Soc.* 110, 281-284, 1990.

[23] Mukhi, S., "SL(2,R) conformal field theory, minimal models and two dimensional gravity", *Proceedings of the International Colloquium on Modern Quantum Field Theory*, *TIFR*, Bombay, India.

[24] Gatto,R., Sartori, G., "Zeros of the D-term and complexification of the gauge group in supersymmetric theories", *Physics Letters B*, 157(5–6), 389-392, 1985.

[25] Richardson, R. W., "Affine Coset Spaces of Reductive Algebraic Groups", *Bulletin of the London Mathematical Society*, 9(1), 38-41, 1977.

[26] Ozeki, I., "On the microlocal structure of the regular prehomogeneous vector space", *Proc. Japan Acad. Ser. A Math. Sci.*, 55(2), 37-40, 1979.

[27] Kirillov, S. A., Kuznetsov, M. I., Chebochko, N. G., "On Deformations of the Lie Algebra of the Type G2 of Characteristic 3", *Russian Mathematics*, 44(3), 31-36, 2000.

[28] Guralnick, R. M., "Invertible Preservers and Algebraic Groups II: Preservers of Similarity Invariants and Overgroups of PSLn (F)", *Linear and Multilinear Algebra*, 43(1-3), 221-255, 1997.

[29] Hinrichsen, D., O'Halloran, J., "A Complete Characterization of Orbit Closures of Controllable Singular Systems under Restricted System Equivalence", *SIAM J. Control Optim.*, 28(3), 602–623, 1990.

[30] Incesu, M., The complete system of point invariants in the similarity geometry, *PhD Thesis*, Karadeniz Technical University, Trabzon, 2008.

[31] Incesu, M., Gursoy, O., ”LS (2)-Equivalence conditions of control points and application to planar Bezier curves”, *New Trends in Mathematical Sciences*, 5(3), 70-84, 2017.