Modelling & Application & Theory
ISSN: 2548–0596
Vol. 2, No. 1, 2017, pp. 38-46

Maximum Principle For The Damped Boussinesq Equation

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Abstract. In this paper, the optimal distributed control problem of an Damped Boussinesq equation is studied. Dubovitskii-Milyutin functional analytical approach is formulated for obtaining the necessary optimality condition in the form of Maximum principle for the system under consideration. Necessary optimality condition is presented in fixed final horizon case.

Keywords. Boussinesq equation, Maximum principle, Deflection, Performance index.

2000 MSC. Primary 49J20, 74H45, 34K35, 47N70.

1. Introduction

Boussinesq equation is introduced by Joseph Boussinesq in 1870s for modelling the propagation of long waves on the surface of water with a small amplitude. Over the last two decades, Boussinesq equation is studied in various aspects by different researchers. In [2], Li considered the maxiumum principle for an optimal control problem governed by Boussinesq equations including integral type state constraints. Analysis and approximation of linear feedback control problems for the Boussinesq equations are studied in [3]. In [5], Wazwaz investigated the logarithmic-Boussinesq equation for Gaussian solitary waves and derived the Gaussian solitary wave solutions for the logarithmic-regularized Boussinesq equation. In [6], Shakhmurov obtained the existence and uniqueness of solution of the integral boundary value problem for abstract Boussinesq equations. Global well-posedness and long time decay of the 3D Boussinesq equations presented in [7]. Also, small global solutions to the damped two-dimensional Boussinesq equations obtained in [8]. The results on local well-posedness for the sixth-order Boussinesq equation are derived in [9]. In [10], damped infinite energy solutions of the 3D Euler and Boussinesq equations are presented. In [11], extended Boussinesq model to predict the propagation of waves in porous media is developed. The inertial and drag resistances are taken into account in the developed model. In [12], Fourier spectral approximation for the time fractional Boussinesq equation with periodic boundary condition is considered. In the light of [1]-[4], this paper is concerned with the optimal distributed control problem for an Damped Boussinesq equation. By adopting Dubovitskii and Milyutin functional analytical approach, necessary optimality condition is obtained in the form of Maximum principle.

2. Well-posedness of the system

Let us consider the following damped Boussinesq equation[13]

$$u_{tt} + \Delta^2 u - \Delta u_t - \Delta f(u) = C(x,t), \quad 0 < x < \ell, \quad 0 < t < T \tag{1}$$

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where state variable u is the displacement function, f is a nonlinear function satisfying specific conditions given in the next , $C(x,t)$ is the control function and $\Delta u = u_{xx}$. Eq.(1) is subject to the following boundary conditions

$$u(0,t) = u(\ell,t) = 0, \quad u_{xx}(0,t) = u_{xx}(\ell,t) = 0 \tag{2}$$

and initial conditions

$$u(x,0) = u_0(x) \in H_0^2(0,\ell), \quad u_t(x,0) = u_1(x) \in L^2(0,\ell). \tag{3}$$

In [14], by using relaxed assumption on the non-linearity in the stiffness constitute law, a definition of weak solution and the existence and uniqueness of such weak solutions are presented. In the light of the [14], the well-posedness of the system given by Eqs.(1)-(3) is shown as follows;

Let $H = L^2(0,\ell)$ and $V = H_0^2(0,\ell)$, so the Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$ with $V^* = H^{-2}(0,\ell)$. Here $\langle .,. \rangle$ is the inner product in H and $\langle .,. \rangle_{V^*,V}$ denotes the usual duality product. Denote $\|.\|_S$ the norm of the space S. Define the space of solutions to be

$$S(0,T) = \{p | p \in L^2(0,T;V), p_t \in L^2(0,T;V), p_{tt} \in L^2(0,T;V^*)\}$$

with norm

$$\|p\|_{S(0,T)} = (\|p\|^2_{L^2(0,T;V)} + \|p_t\|^2_{L^2(0,T;V)} + \|p_{tt}\|^2_{L^2(0,T;V^*)})^{1/2}.$$

A function $u \in S(0,T)$ is weak solution of Eq.1 if the following equation holds:

$$\langle u_{tt}(t), \phi \rangle_{V^*,V} + \langle u_{xx}(t), \phi_{xx} \rangle - \langle u_t(t), \phi_{xx} \rangle - \langle f(u(t)), \phi \rangle = \langle C(t), \phi \rangle_{V^*,V}, \quad \forall \phi \in V, \tag{4}$$

and $u(0) = u_0 \in V$ and $u_t(0) = u_1 \in H$. Assume that the nonlinear function f satisfies the following local Lipschitz condition. Let $B_r(0)$ denote the ball radius r centred at 0 in H, and for some constant L_{B_r}, we have

$$\|f(\xi) - f(\sigma)\|_H \leq L_{B_r} \|\xi - \sigma\|_H, \quad \forall \xi, \sigma \in B_r(0). \tag{5}$$

Then, if the non-linear function f satisfies the local Lipschitz condition given by Eq.(5) and the control function $C \in L^2(0,\ell : V^*)$, there exists a T^* such that Eq.(1) has a unique weak solution on the interval $[0,T^*]$. Assuming that $f \in W^{1,2}(0,T;H)$, which is the Sobolev space defined in [15], the global existence of unique weak solution is guaranteed. Besides, if the non-linear function f is a bounded function, this weak solution is a global solution to the system defined by Eqs.(1)-(3).

3. Optimal control problem formulation

Let $C(x,t) = \alpha(x,t)\beta(t)$, in which $\alpha(x,t) \in L^2(0,T;V^*)$ and the control $\beta(t) \in L^2(0,T)$ distributed over the space-time domain. Then, we can define the performance index of the system given by Eqs.(1)-(3) as follows;

$$\min_{\beta(.) \in U_{ad}} \mathscr{J}(u,\beta) = \min_{\beta(.) \in U_{ad}} \int_0^T \int_0^\ell L(u(x,t), \beta(t), x, t)\, dx dt \tag{6}$$

and the admissible control function constraint is defined by

$$U_{ad} = \{\beta \in L^2(0,T) \mid 0 \leq \beta_0 \leq \beta(t) \leq \beta_1, \quad t \in [0,T] \quad \text{a.e.}\}. \tag{7}$$

The performance index of the system L is quite general in the sense that it contains most practically concerned cost functional like quadratic cost functional of the following form:

$$\mathscr{J}(u,\beta) = \int_0^T \int_0^\ell \mu_1 \mid u(x,t) - u^*(x,t) \mid^2 dxdt + \mu_2 \int_0^T \mid \beta(t) - \beta^*(t) \mid^2 dt,$$

where $\mu_i > 0, \quad i = 1,2,$ are constants and $u^*(x,t), \beta^*(t)$ are, respectively, the optimal control goal for the displacement of vibrating beam and the optimal control force. Take $u \in S(0,T)$. The control space is $L^2(0,T)$ and the control constraint is $\beta_0 \le \beta_t \le \beta_1$ in $(0,T)$. The following assumptions for the cost functional are assumed:

L is a functional defined on $V \times [\beta_0, \beta_1] \times [0,T] \times [0,\ell]$ and

$$\frac{\partial L(u(x,t),\beta(t),x,t)}{\partial u}, \quad \frac{\partial L(u(x,t),\beta(t),x,t)}{\partial \beta}$$

exists for every $(u,\beta) \in V \times [\beta_0, \beta_1]$ and L is continuous in its variables.Also,

$$\int_0^\ell \mid \frac{\partial L(u(x,t),\beta(t),x,t)}{\partial u} \mid dx, \quad \int_0^\ell \mid \frac{\partial L(u(x,t),\beta(t),x,t)}{\partial \beta} \mid dx$$

are bounded for $t \in [0,T]$.

Define $X_T = S(0,T) \times L^2(0,T)$. Let (u^*, β^*) be the optimal solution to the control problem (6) subject to the state equation Eq.(1). Set

$$\Omega_1 = \{(u,\beta) \in X_T \mid \beta_0 \le \beta(t) \le \beta_1, t \in [0,T] \quad \text{a.e.}\},$$

$$\begin{aligned} \Omega_2 = \{(u,\beta) \in X_T \mid\ & u_{tt} + \Delta^2 u - \Delta u_t - \Delta f(u) = \alpha(x,t)\beta(t), u(0,t) = u(\ell,t) = 0, \\ & u_{xx}(0,t) = u_{xx}(\ell,t) = 0, u(x,0) = u_0(x), u_t(x,0) = u_1(x), u(x,T) = u^*(x,T)\}. \end{aligned} \tag{8}$$

Then the optimal control problem Eq.(6) is equivalent to questing for $(u^*, \beta^*) \in \Omega = \Omega_1 \cap \Omega_2$ such that

$$\mathscr{J}(u^*, \beta^*) = \min_{(u,\beta)\in\Omega} \mathscr{J}(u,\beta). \tag{9}$$

It is seen that the problem (9) is an extremum problem on the inequality constraint Ω_1 and the equality constraint Ω_2. In this situation, the functional analytical approach of Dubovitskii and Milyutin has been turned out to be very powerful to solve such kind of extremum problems[16, 17, 18]. The general Dubovitskii-Milyutin theorem for the extremum problem (9) can be stated as the following theorem 1.

Theorem 1. [Dubovitskii-Milyutin] Suppose the functional $\mathscr{J}(u,\beta)$ assumes a minimum at the point (u^*, β^*) in Ω. Assume that $\mathscr{J}(u,\beta)$ is regularly decreasing at (u^*, β^*) with the cone of directions of decrease K_0 and the inequality constraint is regular at (u^*, β^*) with the cone of feasible directions K_1 and that the equality constraint is also regular at (u^*, β^*) with the cone of tangent directions K_2. Then there exists continuous linear functionals f_0, f_1, f_2, not all identically zero, such that $f_i \in K_i^*$, the dual cone of $K_i, \quad i = 0,1,2,$ which satisfy condition

$$f_0 + f_1 + f_2 = 0.$$

4. Pontryagin's maximum principle

4.1. The cone of directions of decrease K_0

In order to apply Theorem 1, we have to determine all cones $K_i, i = 0,1,2$. First, let us find the cone of directions of decrease K_0. By assumption, $\mathscr{J}(u,\beta)$ is differentiable at any point (u^0,β^0) in any direction (u,β) and its directional derivative is

$$\mathscr{J}'(u^0,\beta^0,u,\beta) = \lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}[\mathscr{J}(u^0+\varepsilon u,\beta^0+\varepsilon\beta) - \mathscr{J}(u^0,\beta^0)]$$

$$= \lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\left\{\int_0^T\int_0^\ell [L(u^0+\varepsilon u,\beta^0+\varepsilon\beta,x,t) - L(u^0,\beta^0,x,t)]dxdt\right\}$$

$$= \int_0^T\int_0^\ell \left[\frac{\partial L(u^0,\beta^0,x,t)}{\partial u}u + \frac{\partial L(u^0,\beta^0,x,t)}{\partial \beta}\beta\right]dxdt.$$

Hence, the cone of directions of decrease of the functional $\mathscr{J}(u,\beta)$ at the point (u^*,β^*) is determined by

$$\begin{aligned} K_0 &= \{(u,\beta)\in X_T \mid \mathscr{J}'(u^*,\beta^*;u,\beta) < 0\} \\ &= \left\{(u,\beta)\in X_T \mid \int_0^T\int_0^\ell \left[\frac{\partial L(u^*,\beta^*,x,t)}{\partial u}u + \frac{\partial L(u^*,\beta^*,x,t)}{\partial \beta}\beta\right]dxdt < 0\right\}. \end{aligned} \tag{10}$$

If $K_0 \neq \emptyset$, then for any $f_0 \in K_0^*$, there exists a $\kappa_0 \geq 0$ such that

$$f_0(u,\beta) = -\kappa_0\int_0^T\int_0^\ell \left[\frac{\partial L(u^*,\beta^*,x,t)}{\partial u}u + \frac{\partial L(u^*,\beta^*,x,t)}{\partial \beta}\beta\right]dxdt\bigg\}. \tag{11}$$

4.2. The cone of feasible directions K_1

Since $\Omega_1 = S(0,T)\times\tilde{\Omega}_1$, in which $\tilde{\Omega}_1 = \{\beta\in L^2(0,T)|\beta_0 \leq \beta(t) \leq \beta_1, t\in[0,T] \quad \text{a.e.}\}$ is a closed convex subset of $L^2(0,T)$, the interior $\dot{\Omega}_1$ of Ω_1 is not empty, and at point (u^*,β^*), the cone of feasible directions K_1 of Ω_1 is determined by

$$K_1 = \{\kappa(\dot{\Omega}_1 - (u^*,\beta^*))|\kappa > 0\} = \{h|h = \kappa(u-u^*,\beta-\beta^*),(u,\beta)\in\dot{\Omega},\kappa > 0\}. \tag{12}$$

Therefore, for any arbitrary $f_1 \in K_1^*$, if there is an $\bar{a}(t)\in L^2(0,T)$, such that the linear functional defined by

$$f_1(u,\beta) = \int_0^T \bar{a}(t)\beta(t)dt \tag{13}$$

is a support to $\tilde{\Omega}_1$ at point β^*, then

$$\bar{a}(t)[\beta(t)-\beta^*] \geq 0, \quad \forall\beta(t)\in[\beta_0,\beta_1], \quad t\in[0,T] \quad \text{a.e.} \tag{14}$$

4.3. The cone of tangent directions K_2

Let $\chi(x,t) = u_{tt} + \Delta^2 u - \Delta u_t - \Delta f(u) - \alpha(x,t)\beta(t)$. Define the operator $G: X_T \to L^2(0,T;V^*) \times (L^2(0,T))^4 \times V \times H \times V$ by

$$G(u,\beta) = (\chi(x,t), u(0,t), u(\ell,t), u_{xx}(0,t), u_{xx}(0,\ell), u(x,0) - u_0(x), u_t(x,0) - u_1(x), u(x,T) - u^*(x,T)). \tag{15}$$

Then

$$\Omega_2 = \{(u,\beta) \in X_T \mid G(u(x,t),\beta(t)) = 0\}, \tag{16}$$

so the Fréchet derivative of the operator $G(u,\beta)$ is

$$G'(u,\beta)(\hat{u},\hat{\beta}) = (\hat{\chi}(x,t), \hat{u}(0,t), \hat{u}(\ell,t), \hat{u}_{xx}(0,t), \hat{u}_{xx}(0,\ell), \hat{u}(x,0), \hat{u}_t(x,0), \hat{u}(x,T))$$

in which $\hat{\chi}(x,t) = \hat{u}_{tt}(x,t) + \Delta^2\hat{u}(x,t) - \Delta\hat{u}_t(x,t) - \Delta[f'(u(x,t))\hat{u}(x,t)] - \alpha(x,t)\hat{\beta}(x,t)$. Since u^*, β^* is the solution to the problem Eq.(6), it has $G(u^*,\beta^*) = 0$. Choosing arbitrary

$$(g_1,g_2,g_3,g_4,g_5,g_6,g_7,g_8) \in L^2(0,T;V^*) \times (L^2(0,T))^4 \times V \times H \times V$$

and solving the equation

$$G'(u^*,\beta^*)(\hat{u},\hat{\beta}) = (g_1(x,t), g_2(t), g_3(t), g_4(t), g_5(t), g_6(x), g_7(x), g_8(x)),$$

we obtain

$$\begin{aligned} \hat{u}_{tt}(x,t) + \Delta^2\hat{u}(x,t) - \Delta\hat{u}_t(x,t) - \Delta[f'(u^*(x,t))\hat{u}(x,t)] = \alpha(x,t)\hat{\beta}(t) + g_1(x,t), u(0,t) = g_2(t), \\ u(\ell,t) = g_3(t), u_{xx}(0,t) = g_4(t), u_{xx}(\ell,t) = g_5(t), u(x,0) = g_6(x), u_t(x,0) = g_7(x), u(x,T) = g_8(x). \end{aligned} \tag{17}$$

Next, assume that the linearized system

$$\begin{aligned} u_{tt}(x,t) + \Delta^2 u(x,t) - \Delta u_t(x,t) - \Delta[f'(u^*(x,t))u(x,t)] = \alpha(x,t)\beta(t), \\ u(0,t) = 0, u(\ell,t) = 0, u_{xx}(0,t) = 0, u_{xx}(\ell,t) = 0, u(x,0) = 0, u_t(x,0) = 0 \end{aligned} \tag{18}$$

is controllable. Then choose $\beta(t) = \hat{\beta}(t) \in L^2(0,T)$ such that $u(x,T) = g_8(x) - \eta(x,T)$ and let u be the solution to the linearized system (18). Choose $\hat{u}(x,t) = u(x,t) + \eta(x,t)$, where η satisfies the following equation:

$$\begin{aligned} \eta_{tt}(x,t) + \Delta^2\eta(x,t) - \Delta\eta_t(x,t) - \Delta[f'(u^*(x,t))\eta(x,t)] = g_1(x,t), \eta(0,t) = g_2(t), \eta(\ell,t) = g_3(t), \\ \eta_{xx}(0,t) = g_4(t), \eta_{xx}(\ell,t) = g_5(t), \eta(x,0) = g_6(x), \eta_t(x,0) = g_7(x), \eta(x,T) = g_8(x). \end{aligned} \tag{19}$$

In this way, it suffices for $(\hat{u},\hat{\beta})$ satisfying (17). Therefore, $G'(u^*,\beta^*)$ maps X_T on to $L^2(0,T;V^*) \times (L^2(0,T))^4 \times V \times H \times V$. Moreover, the cone of the tangent directions K_2 to the constraint Ω_2 at the point (u^*,β^*) consists of the kernel of $G'(u^*,\beta^*)$, i.e. (u,β) satisfies the following equation in X_T :

$$\begin{aligned} u_{tt}(x,t) + \Delta^2 u(x,t) - \Delta u_t(x,t) - \Delta[f'(u^*(x,t))u(x,t)] = \alpha(x,t)\beta(t), \\ u(0,t) = 0, u(\ell,t) = 0, \quad u_{xx}(0,t) = 0, u_{xx}(\ell,t) = 0, u(x,0) = 0, u_t(x,0) = 0 \end{aligned} \tag{20}$$

and

$$u(x,T) = 0. \tag{21}$$

Define

$$\begin{aligned} K_{21} = \{(u,\beta) \in X_T \mid (u(x,t),\beta(t)) \quad \text{satisfies (20)}\}, \\ K_{22} = \{(u,\beta) \in X_T \mid (u(x,t),\beta(t)) \quad \text{satisfies (21)}\}. \end{aligned}$$

Then the cone of tangent directions $K_2 = K_{21} \cap K_{22}$. Hence,

$$K_2^* = K_{21}^* + K_{22}^*.$$

For any $f_2 \in K_2^*$, decompose $f_2 = f_{21} + f_{22}, f_{2i} \in K_{2i}^*$, the dual cone of $K_{2i}, \quad i = 1,2$. Then $f_{21}(u,\beta) = 0$ and for all $u(x,t) \in S(0,T)$ satisfying $u(x,T) = 0$, there exists a $\varphi(x) \in V^*$ such that

$$f_{22}(u(x,t),\beta(t)) = \int_0^\ell u(x,t)\varphi(x)dx.$$

It then follows from Theorem 1 that there exist continuous linear functionals, not all identically zero, such that

$$f_0 + f_1 + f_{21} + f_{22} = 0.$$

Therefore, when selecting (u,β) satisfying (20), $f_{21}(u,\beta) = 0$. Moreover,

$$\begin{aligned} f_1(u(x,t),\beta(t)) &= -f_0(u(x,t),\beta(t)) - f_{22}(u(x,t),\beta(t)) \\ &= \kappa_0 \int_0^T \int_0^\ell \left[\frac{\partial L(u^*,\beta^*,x,t)}{\partial u} u(x,t) + \frac{\partial L(u^*,\beta^*,x,t)}{\partial \beta} \beta(t) \right] dxdt - \int_0^\ell u(x,T)\varphi(x)dx. \end{aligned} \tag{22}$$

4.4. Maximum principle of problem (6)

Define the adjoint system of the linearized system (18)

$$\begin{aligned} &\phi_{tt}(x,t) + \Delta^2\phi(x,t) - \Delta\phi_t(x,t) - \Delta[f'(u^*(x,t))\phi(x,t)] = \kappa_0 \frac{\partial L(u^*,\beta^*,x,t)}{\partial u}, \\ &\phi(0,t) = 0, \phi(\ell,t) = 0, \quad \phi_{xx}(0,t) = 0, \phi_{xx}(\ell,t) = 0, \phi(x,T) = \mu(x) \end{aligned} \tag{23}$$

with

$$\mu(x) = \begin{cases} \int_0^x \int_0^{x_3} \int_0^{x_2} \int_0^{x_1} \varphi(s)dsdx_1dx_2dx_3, & x \neq \ell, \\ 0, & x = \ell. \end{cases} \tag{24}$$

As with (1), the existence and uniqueness of the solution to (23) can be obtained similarly.

Theorem 2. The solution of system (18) and that of its adjoint system (23) have the following relation:

$$\kappa_0 \int_0^T \int_0^\ell \frac{\partial L(u^*,\beta^*,x,t)}{\partial u} u(x,t)dxdt - \int_0^\ell u(x,T)\varphi(x)dx = \int_0^T \int_0^\ell \alpha(x,t)\beta(t)\phi(x,t)dxdt \tag{25}$$

Proof. Multiply the first equation in (23) by $u(x,t)$ and integrate it by parts over $[0,T] \times [0,\ell]$ with respect to t and x, respectively. The proof then follows. ■

Now, by virtue of Theorem 2, we can rewrite $f_1(u,\beta)$ as

$$f_1(u,\beta) = \int_0^T \left\{ \int_0^\ell \left[\kappa_0 \frac{\partial L(u^*,\beta^*,x,t)}{\partial \beta} + \alpha(x,t)\phi(x,t) \right] dx \right\} \beta(t)dt. \tag{26}$$

Therefore,

$$\bar{a}(t) = \int_0^\ell \left[\kappa_0 \frac{\partial L(u^*,\beta^*,x,t)}{\partial \beta} + \alpha(x,t)\phi(x,t) \right] dx$$

and (14) then reads as

$$\left\{\int_0^\ell \left[\kappa_0 \frac{\partial L(u^*,\beta^*,x,t)}{\partial \beta} + \alpha(x,t)\phi(x,t)\right]dx\right\}.[\beta(t)-\beta^*(t)] \geq 0, \tag{27}$$
$$\forall \beta(t) \in [\beta_0,\beta_1], \quad t \in [0,T] \quad \text{a.e., and } \phi(x,t) \neq 0, \kappa_0 \neq 0.$$

Since otherwise, there are definitely $f_0 = 0, f_1 = 0, f_{22} = 0$ and $f_{21} = 0$, which contradict with the fact in Theorem 1 that these continuous linear functionals are not all identically zero. On the other hand, if K_0 is a null set, then there is

$$\int_0^T \int_0^\ell \left[\frac{\partial L(u^*,\beta^*,x,t)}{\partial u} u(x,t) + \frac{\partial L(u^*,\beta^*,x,t)}{\partial \beta}\beta(t)\right]dxdt = 0, \quad \forall (u,\beta) \in X_T.$$

In particular, if we choose $\kappa_0 = 1$ and $\varphi(x) = 0$, it follows from Theorem 2 that

$$\int_0^T \int_0^\ell \frac{\partial L(u^*,\beta^*,x,t)}{\partial u} u(x,t)dxdt = \int_0^T \int_0^\ell \alpha(x,t)\phi(x,t)\beta(t)dxdt.$$

Therefore,

$$\int_0^T \left\{\int_0^\ell \left[\frac{\partial L(u^*,\beta^*,x,t)}{\partial \beta} + \alpha(x,t)\phi(x,t)\right]dx\right\}\beta(t)dt = 0, \quad \forall \beta(t) \in L^2(0,T),$$

from which we obtain

$$\int_0^\ell \left[\frac{\partial L(u^*,\beta^*,x,t)}{\partial \beta} + \alpha(x,t)\phi(x,t)\right]dx = 0, \quad \forall t \in [0,T] \quad \text{a.e.}$$

Therefore, (27) still holds. Finally, if there is a nonzero solution to the adjoint system

$$\hat{\phi}_{tt} + \Delta^2\hat{\phi} - \Delta\hat{\phi}_t - \Delta[f'(u^*(x,t))\hat{\phi}(x,t)] = \kappa_0 \frac{\partial L(u^*,\beta^*,x,t)}{\partial u},$$
$$\hat{\phi}(0,t) = 0, \hat{\phi}(\ell,t) = 0, \quad \hat{\phi}_{xx}(0,t) = 0, \hat{\phi}_{xx}(\ell,t) = 0, \hat{\phi}(x,T) = \mu(x) \tag{28}$$

such that the following equality holds true

$$\int_0^\ell \alpha(x,t)\hat{\phi}(x,t)dx = 0, \quad \forall t \in [0,T] \quad \text{a.e.,}$$

then when we choose $\kappa_0 = 0, \quad \mu(x) = \hat{\phi}(x,T)$, which is defined as (24), (27) is still valid. Since otherwise, if for any nonzero solution $\hat{\phi}$ of (28), it has

$$\int_0^\ell \alpha(x,t)\hat{\phi}(x)dx \neq 0, \tag{29}$$

in this case, we say that the situation is non-degenerate. Then the linearized system (18) is controllable. In fact, if (18) is not controllable, then there exists a $\varphi(x) \in V^*$ such that

$$\int_0^\ell u(x,T)\varphi(x)dx = 0, \quad \varphi(x) \neq 0.$$

Choose $\kappa_0 = 0$ and $\hat{\phi}$ to be the solution of (28). Then it follows from Theorem 2 that

$$\int_0^T \int_0^\ell \alpha(x,t)\hat{\phi}(x,t)\beta(t)dxdt = 0, \forall \beta(t) \in L^2(0,T).$$

Hence,

$$\int_0^\ell \alpha(x,t)\hat{\phi}(x,t)dx = 0, \quad \forall t \in [0,T] \quad \text{a.e.}$$

This is a contradiction. Therefore, under the case of (28), the system (29) is controllable. Combining the results above, we have obtained the Pontryagin's maximum principle for the optimal control problem (6) subject to the system (1).

Theorem 3. Suppose (u^*,β^*) is a solution to the optimal control problem (6). Then there exist $\kappa_0 \geq 0$ and $\phi(x,t) \neq 0$, such that the following maximum principle holds true:

$$\left\{\int_0^\ell \left[\kappa_0 \frac{\partial L(u^*,\beta^*,x,t)}{\partial \beta} + \alpha(x,t)\phi(x,t)\right] dx\right\}.\left[\beta(t) - \beta^*(t)\right] \geq 0,$$

$$\forall \beta(t) \in [\beta_0,\beta_1], t \in [0,T] \quad \text{a.e.},$$

where the function $\phi(x,t)$ satisfies the adjoint equation (23).

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