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Oscillation Of Higher-Order Half-Linear Dynamic Equations On Time Scales

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Abstract. In this paper we are concerned with the oscillation of solutions of higher-order half-linear dynamic equations of the form

$$\left(p(t)\left(\left(x(t)+q(t)x(\tau(t))\right)^{\Delta^{m-1}}\right)^{\alpha}\right)^{\Delta}+r(t)x^{\beta}(\varphi(t))=0.$$

The main theorem improves some results for the corresponding higher-order difference and differential equations.

Keywords. Time scale, higher-order half-linear dynamic equation.

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1. Introduction

The oscillation theory of delay differential and difference equations play an important role in many areas of mathematics such as technology, economics, mathematical biology. The theory of time scales unifies continuous and discrete analysis. Therefore the study of dynamic equations on time scales is an area of mathematics that recently has received a lot of attention. We refer the reader to book, by Bohner and Peterson [3] for a comprehensive treatment of the subject on the subject of time scale.

During the last few decades, there has been extensive improvement in the oscillation and nonoscillation theory of delay difference/differential/dynamic equations.

In the paper by Han, Sun, Zhang and Li [7], some new oscillation criterias for the second order nonlinear delay dynamic equation

$$\left(p(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+q(t)f(x(\tau(t)))=0$$

was established.

Grace et al. [6], studied the oscillation of nth order nonlinear dynamic equations of the form

$$x^{\Delta^{n}}(t)+q(t)\left(x^{\sigma}(\xi(t))\right)^{\lambda}=0.$$

Grace [5], presented some new criterias for the oscillation of even order dynamic equation

$$\left(a(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta}+q(t)(x(t))^{\alpha}=0$$

on time scale under the condition $\int_{t_0}^{\infty}a^{-\frac{1}{\alpha}}(s)\Delta s=\infty$.

In this paper we study the oscillation of higher-order half-linear dynamic equations of the form

$$\left(p(t)\left(\left(x(t)+q(t)x(\tau(t))\right)^{\Delta^{m-1}}\right)^{\alpha}\right)^{\Delta}+r(t)x^{\beta}(\varphi(t))=0,\ t\geq t_0 \tag{1}$$

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where we assume

$$\int_{t_0}^{\infty} p^{-\frac{1}{\alpha}}(t)\,\Delta t < \infty \tag{2}$$

without using $p^{\Delta}(t) > 0$.

Throughout this paper, we assume $\alpha,\ \beta$ are the ratio of odd positive integers, $\beta \leq \alpha,\ p(t) \in C^1_{rd}[t_0,\infty)_{\mathbb{T}},\ p(t) > 0,\ q(t)$ is an oscillating function satisfying $\lim_{t\to\infty} q(t) = 0,\ r(t),\ \tau(t),\ \varphi(t) \in C_{rd}[t_0,\infty)_{\mathbb{T}},\ r(t) > 0$ with $r^{\Delta}(t) > 0$ for $t \geq t_0,\ \tau(t) < t$ and $\lim_{t\to\infty} \tau(t) = \infty,\ \varphi(t) < t$ and $\lim_{t\to\infty} \varphi(t) = \infty$. We assume that the solution of Eq.(1) is the function $x(t) \in C^{n-1}_{rd}[T_x,\infty)_{\mathbb{T}},\ T_x \geq t_0$. We consider those solutions $x(t)$ of Eq.(1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume the reader is familiar with the notation and basic results for dynamic equations on time scales. For a review of this topic we direct the reader to the monograph [3].

2. Main Results

We need the following lemmas in order to obtain our main results. The second one is the well-known Kiguradze's lemma.

Lemma 1. [3]Let u and v be continuous functions on $[a,b]$ that are $\Delta-$differentiable on $[a,b)$. If u^{Δ} and v^{Δ} are integrable from a to b, then

$$\int_a^b u^{\Delta}(t)\,v(t)\,\Delta(t) + \int_a^b u^{\sigma}(t)\,v^{\Delta}(t)\,\Delta(t) = u(b)\,v(b) - u(a)\,v(a).$$

Lemma 2. [2] Let $x \in C^m_{rd}([t_0,\infty),\mathbb{R}^+)$. If $x^{\Delta^m}(t)$ is of constant sign on $[t_0,\infty)_{\mathbb{T}}$ and not identically zero on $[t_1,\infty)_{\mathbb{T}}$ for any $t_1 \geq t_0$, then there exists a $t_x \geq t_0$ and and an integer $l,\ 0 \leq l \leq m$ with $m+l$ even for $x^{\Delta^m}(t) \geq 0$, or $m+l$ odd for $x^{\Delta^m}(t) \leq 0$ such that

$$l > 0 \text{ implies } x^{\Delta^k}(t) > 0 \text{ for } t \geq t_x,\ k \in \{1,2,...,l-1\}$$

and

$$l \leq m-1 \text{ implies } (-1)^{l+k} x^{\Delta^k}(t) > 0 \text{ for } t \geq t_x,\ k \in \{l,l+1,...,m-1\}.$$

Lemma 3. [8]Let f be n-times differentiable on $\mathbb{T}^{\kappa^n}$. If $f^{\Delta} > 0$, then for every $\lambda,\ 0 < \lambda < 1$, we have

$$f(t) \geq \lambda\,(-1)^{n-1}\,g_{n-1}(\sigma(T^*),t)\,f^{\Delta^{n-1}}(t).$$

Lemma 4. If the inequality

$$x^{\Delta}(t) + q(t)\,x^{\beta}(\tau(t)) \leq 0 \tag{3}$$

has an eventually positive (negative) solution, then the equation

$$x^{\Delta}(t) + q(t)\,x^{\beta}(\tau(t)) = 0$$

also has an eventually positive (negative) solution where q and β are as in Eq.(1).

Proof. Let $x(t)$ be an eventually positive solution of Eq.(3). One can easily see that $x^{\Delta}(t) < 0$ for all $t \geq t_0$ so that $x(t)$ is decreasing. Integrating the Eq.(3) from $t \geq t_0$ to $u \geq t$ and letting $u \to \infty$, we have

$$x(t) \geq \int_t^{\infty} q(s)\,x^{\beta}(\tau(s))\,\Delta s.$$

Define

$$\Phi(t,x(\tau(t))) := \int_{t_0}^{\infty} q(s)\,x^{\beta}(\tau(s))\,\Delta s - c$$

where $c = x(t_0)$. Now we show that the existence of a positive solution to the equation

$$y(t) = c + \Phi(t, y(\tau(t))) \;\; for\, t \geq t_0.$$

In order to do this, we define the function sequence $\{y_k(t)\}$, $k = 0, 1, ...$ such that

$$y_0(t) = x(t),\; y_{k+1}(t) = c + \Phi(t, y_k(\tau(t))) \;\; for\, t \geq t_0. \tag{4}$$

Then, one can easily see that $y_k(t)$ is well-defined and

$$0 \leq y_k(t) \leq x(t),\; c \leq y_{k+1}(t) \leq y_k(t).$$

Thus, the sequence $\{y_k\}$ is positive and nonincreasing in k for $t \geq t_0$. So that we can define $y(t) = \lim_{k\to\infty} y_k(t)$. Since $0 < y(t) \leq y_k(t) \leq x(t)$ for all $k \geq 0$ and

$$\Phi(t, y_k(\tau(t))) \leq \Phi(t, x(\tau(t))),$$

the convergence of $\{y_k(t)\}$ is uniform with respect to t. Then taking the limit of both sides of (4), we obtain

$$y(t) = c + \Phi(t, y(\tau(t))) \;\; for\, t \geq t_0. \tag{5}$$

Finally taking the delta derivative of the Eq.(5), we have

$$y^{\Delta}(t) + q(t)\,y^{\beta}(\tau(t)) = 0.$$

Hence, Eq.(3) has a positive solution $x(t)$. This completes the proof. ■

For the sake of convenience, we define the function

$$z(t) := x(t) + q(t)\,x(\tau(t)). \tag{6}$$

Theorem 1. Let $m \geq 2$. Assume that (2) holds and for some constant $\lambda_0 \in (0,1)$, the dynamic equation

$$y^{\Delta}(t) + r(t)\left(\frac{\lambda h_{m-1}(\varphi(t), \sigma(T^*))}{p^{\frac{1}{\alpha}}(\tau(t))}\right)^{\beta} y^{\frac{\beta}{\alpha}}(\varphi(t)) = 0 \tag{7}$$

is oscillatory. If

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[M^{\beta-\alpha} r(s)\,\lambda^{\beta}\,\delta^{\alpha}(s)\,h_{m-2}^{\beta}(\tau(s), \sigma(T^*)) + \frac{(\delta^{\alpha})^{\Delta}(s)}{\delta^{\alpha}(s)} \right] \Delta s = \infty \tag{8}$$

holds for every constant $M > 0$ where

$$\delta(t) := \int_{t}^{\infty} p^{-\frac{1}{\alpha}}(s)\,\Delta s,$$

then every solutions of Eq.(1) is oscillatory or tends to zero.

Proof. Assume that Eq.(1) has a bounded non-oscillatory solution $x(t)$. Without loss of generality, we assume that $x(t)$ is eventually positive (the proof is similar in the case $x(t)$ is eventually negative). Moreover suppose that $\lim_{t\to\infty} x(t) \neq 0$.

Using Eq.(1) and (6), we have

$$\left(p(t)\left(z^{\Delta^{m-1}}(t)\right)^{\alpha}\right)^{\Delta} = -r(t)x^{\beta}(\varphi(t)) \leq 0,\ t \geq t_1. \tag{9}$$

Since we assume that $x(t)$ is bounded and $\lim_{t\to\infty} x(t) \neq 0$, we have $\lim_{t\to\infty} q(t)x(\tau(t)) = 0$. Then there is a $t_2 \geq t_1$ such that $z(t) = x(t) + q(t)x(\tau(t)) > 0$ eventually. So that $z(t)$ is also bounded for sufficiently large $t \geq t_2$.

Using lemma 2, we can see that there exist two possible cases:

1) $z(t) > 0,\ z^{\Delta^{m-1}}(t) > 0,\ z^{\Delta^{m}}(t) < 0,\ and\ \left(p(t)\left(z^{\Delta^{m-1}}(t)\right)^{\alpha}\right)^{\Delta} < 0,$

2) $z(t) > 0,\ z^{\Delta^{m-2}}(t) > 0,\ z^{\Delta^{m-1}}(t) < 0,\ and\ \left(p(t)\left(z^{\Delta^{m-1}}(t)\right)^{\alpha}\right)^{\Delta} < 0,$

for all $t \geq t_2$, t_2 is large enough. By using (6), there exists a $t_3 \geq t_2$, t_2 is large enough. So that we obtain

$$x(t) = z(t) - q(t)x(\tau(t)) \geq \frac{1}{2}z(t) > 0$$

for all $t \geq t_3$. Since $x(\varphi(t)) > 0$, we have

$$x(\varphi(t)) \geq \frac{1}{2}z(\varphi(t)) > 0. \tag{10}$$

In view of (9) and (10), we obtain

$$\left(p(t)\left(z^{\Delta^{m-1}}(t)\right)^{\alpha}\right)^{\Delta} + \frac{1}{2}r(t)z^{\beta}(\varphi(t)) \leq 0$$

for $t \geq t_3$. Assume that case (1) holds. From lemma 3, we have

$$x(t) \geq \frac{\lambda h_{m-1}(t,\sigma(T^*))x^{\Delta^{m-1}}(t)\,p^{\frac{1}{\alpha}}(t)}{p^{\frac{1}{\alpha}}(t)}$$

for every $\lambda \in (0,1)$ and for all sufficiently large t. Hence by Eq.(1), we see that

$$y(t) := p(t)\left(z^{\Delta^{m-1}}(t)\right)^{\alpha}$$

is a positive solution of the difference inequality

$$y^{\Delta}(t) + \frac{1}{2}r(t)\left(\frac{\lambda h_{m-1}(\varphi(t),\sigma(T^*))}{p^{\frac{1}{\alpha}}(\tau(t))}\right)^{\beta} y^{\frac{\beta}{\alpha}}(\varphi(t)) \leq 0,\ t \geq t_5.$$

Therefore by lemma 4, Eq.(7) also has a positive solution. This is a contradiction.

Assume that case (2) holds. Define the Riccati substitution

$$w(t) := \frac{p(t)\left(z^{\Delta^{m-1}}(t)\right)^{\alpha}}{\left(z^{\Delta^{m-2}}(t)\right)^{\alpha}},\ t \geq t_1. \tag{11}$$

From the case (2), we get $w(t)<0$ for $t\geq t_1$. Since $p(t)\left(z^{\Delta^{m-1}}(t)\right)^{\alpha}$ is decreasing, we have

$$p^{\frac{1}{\alpha}}(s)\,z^{\Delta^{m-1}}(s)\leq p^{\frac{1}{\alpha}}(t)\,z^{\Delta^{m-1}}(t),\ s\geq t\geq t_1.$$

Dividing the above inequality by $p^{\frac{1}{\alpha}}(s)$ and integrating it from t to l,

$$\int_t^l z^{\Delta^{m-1}}(s)\Delta s\leq\int_t^l\frac{p^{\frac{1}{\alpha}}(t)\,z^{\Delta^{m-1}}(t)}{p^{\frac{1}{\alpha}}(s)}\Delta s$$

we obtain

$$z^{\Delta^{m-2}}(l)\leq z^{\Delta^{m-2}}(t)+p^{\frac{1}{\alpha}}(t)\,z^{\Delta^{m-1}}(t)\int_t^l\frac{1}{p^{\frac{1}{\alpha}}(s)}\Delta s.$$

Letting $l\to\infty$, we get

$$0\leq z^{\Delta^{m-2}}(t)+p^{\frac{1}{\alpha}}(t)\,z^{\Delta^{m-1}}(t)\,\delta(t),$$

which yields

$$-\frac{p^{\frac{1}{\alpha}}(t)\,z^{\Delta^{m-1}}(t)}{z^{\Delta^{m-2}}(t)}\delta(t)\leq 1.$$

Thus, by (11) we can see that

$$-w(t)\,\delta^{\alpha}(t)\leq 1. \tag{12}$$

From (11), we have

$$w^{\Delta}(t)=\frac{\left(p(t)\left(z^{\Delta^{m-1}}(t)\right)^{\alpha}\right)^{\Delta}}{\left(z^{\Delta^{m-2}}(\sigma(t))\right)^{\alpha}}-\frac{p(t)\left(z^{\Delta^{m-1}}(t)\right)^{\alpha}}{\left(z^{\Delta^{m-2}}(t)\right)^{\alpha}\mu(t)}+\frac{p(t)\left(z^{\Delta^{m-1}}(t)\right)^{\alpha}}{\left(z^{\Delta^{m-2}}(\sigma(t))\right)^{\alpha}\mu(t)}.$$

Multiplying the last term of the above equation by $\frac{\left(z^{\Delta^{m-2}}(t)\right)^{\alpha}}{\left(z^{\Delta^{m-2}}(t)\right)^{\alpha}}$, we get

$$w^{\Delta}(t)=\frac{\left(p(t)\left(z^{\Delta^{m-1}}(t)\right)^{\alpha}\right)^{\Delta}}{\left(z^{\Delta^{m-2}}(\sigma(t))\right)^{\alpha}}-\frac{w(t)}{\mu(t)}+\frac{w(t)\left(z^{\Delta^{m-2}}(t)\right)^{\alpha}}{\left(z^{\Delta^{m-2}}(\sigma(t))\right)^{\alpha}\mu(t)}. \tag{13}$$

From the case (2), since $z^{\Delta^{m-2}}(t)>0$ and $z^{\Delta^{m-1}}(t)<0$, $z^{\Delta^{m-2}}(t)$ is decreasing. Therefore $z^{\Delta^{m-2}}(t)\geq z^{\Delta^{m-2}}(\sigma(t))>0$. Hence, we have

$$\frac{w(t)\left(z^{\Delta^{m-2}}(t)\right)^{\alpha}}{\left(z^{\Delta^{m-2}}(\sigma(t))\right)^{\alpha}\mu(t)}\leq\frac{w(t)}{\mu(t)}$$

for all $t\geq t_1$. Using inequality (10), (13) becomes

$$w^{\Delta}(t)\leq-\frac{1}{2}r(t)\,\frac{z^{\beta}(\varphi(t))}{\left(z^{\Delta^{m-2}}(t)\right)^{\alpha}}.$$

In view of lemma 3, we get

$$x(t)\geq\lambda h_{m-2}(t,\sigma(T^*))\,x^{\Delta^{m-2}}(t)$$

for every $\lambda\in(0,1)$ and for all sufficiently large t. We can easily see that

$$z(\varphi(t))\geq\lambda h_{m-2}(\varphi(t),\sigma(T^*))\,z^{\Delta^{m-2}}(\varphi(t)).$$

Then there exists a constant $M > 0$ such that

$$\begin{aligned} w^{\Delta}(t) &\leq -\frac{1}{2}r(t)\frac{\lambda^{\beta}h_{m-2}^{\beta}(\varphi(t),\sigma(T^{*}))\left(z^{\Delta^{m-2}}(\varphi(t))\right)^{\beta}}{\left(z^{\Delta^{m-2}}(t)\right)^{\alpha}} \\ &\leq -\frac{1}{2}r(t)\lambda^{\beta}h_{m-2}^{\beta}(\varphi(t),\sigma(T^{*}))M^{\beta-\alpha} \end{aligned}$$

for $t \geq t_1$. Multiplying the above inequality by $\delta^{\alpha}(t)$ and integrating it from t_2 to t, we obtain

$$\begin{aligned} \frac{1}{2}\int_{t_2}^{t} M^{\beta-\alpha}r(s)\lambda^{\beta}\delta^{\alpha}(s)h_{m-2}^{\beta}(\varphi(t),\sigma(T^{*}))\Delta s &\leq \delta^{\alpha}(t_2)w(t_2)-\delta^{\alpha}(t)w(t) \\ &\quad +\int_{t_2}^{t}(\delta^{\alpha})^{\Delta}(s)w(\sigma(s))\Delta s. \end{aligned}$$

Using (12), we have

$$\int_{t_2}^{t}\left[M^{\beta-\alpha}r(s)\lambda^{\beta}\delta^{\alpha}(s)h_{m-2}^{\beta}(\varphi(t),\sigma(T^{*}))+\frac{(\delta^{\alpha})^{\Delta}(s)}{\delta^{\alpha}(s)}\right]\Delta s \leq \delta^{\alpha}(t_2)w(t_2)+1.$$

From (11) and the fact that $(\delta^{\alpha})^{\Delta}(s) < 0$ we arrive at a contradiction. This completes the proof. ■

Corollary 1. Let $m \geq 2$ and $\int_{t_0}^{\infty} p^{-\frac{1}{\alpha}}(t)\Delta t < \infty$. Moreover assume that $\alpha > \beta$, is strictly increasing function,

$$\limsup_{t\to\infty}\int_{\varphi(t)}^{t} r(s)\frac{h_{m-1}^{\beta}\left(\varphi^{m-1}(s),\sigma(T^{*})\right)}{p^{\frac{\beta}{\alpha}}(\varphi(s))}\Delta s > 0.$$

If (8) holds for some constant $\lambda_1 \in (0,1)$ and for all constant $M > 0$, then every solution of Eq.(1) is oscillatory or tends to zero.

3. Conclusion

In this paper, we have studied higher-order half-linear dynamic equation of the form Eq.(1). To the best of author's observation, the results regarding second, fourth and higher order equations have been established under the assumptions, $p^{\Delta}(t) > 0$, $\int_{t_0}^{\infty} p^{-\frac{1}{\alpha}}(t)\Delta t = \infty$ and $\alpha = \beta$. In this note we have not used these assumptions and we have obtained new results on time scales. The main theorem improves previously results and this presents a new approach.

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