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The Banach Algebra of Functions With Fourier Transforms in Weighted Amalgam Spaces

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Abstract. In this paper, we define $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ to be space of all functions in $\left(L^p_{\vartheta_1},\ell^1\right)$ whose Fourier transforms belong to $\left(L^q_{\vartheta_2},\ell^r\right)$. Moreover, we consider the basic and advance properties of this space including Banach algebra, translation invariant, Banach module, a generalized type of Segal algebra etc. Also, we study some inclusions, compact embeddings in sense to weights and further discuss multipliers of this space.

Keywords. Amalgam spaces, inclusion, compact embedding, multipliers, Fourier transform.

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1. Introduction

Let G be a locally compact abelian group with Haar measure μ. For $1 \leq p,q \leq \infty$, an amalgam space $(L^p,\ell^q)(G)$ is to be a Banach space of all measurable functions on G which belong locally to L^p and globally to ℓ^q. Many authors have considered special cases of amalgam spaces such as Krogstad [22], Liu et al. [26], Szeptycki [35], Wiener [37]. Also, Holland [21] presented an important study for amalgams on the real line. In 1979, Stewart [34] extended Holland's definition to locally compact abelian groups for locally compact groups by the Structure Theorem.

For $1 \leq p < \infty$, the set $A_p\left(\mathbb{R}^d\right)$ is the space of all complex-valued functions in $L^1\left(\mathbb{R}^d\right)$ whose the Fourier transforms belong to $L^p\left(\mathbb{R}^d\right)$. This space have considered some authors, see [23], [24], [27]. Moreover, Feichtinger and Gurkanli [12], Fischer et al. [13] and Gurkanli [16] have found some generalized results in sense to weights.

In 1926, Wiener [37] presented the first systematical work on amalgam spaces (L^p,ℓ^q). The usage areas of the amalgam spaces are generally harmonic and time-frequency analysis. For an another historical journey, we can refer [14]. Moreover, the amalgam spaces or some special cases of these spaces were investigate by a number of authors, see [4], [5], [18], [19].

In this paper, we define a space $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G) = \left\{f \in \left(L^p_{\vartheta_1},\ell^1\right) : \widehat{f} \in \left(L^q_{\vartheta_2},\ell^r\right)\right\}$ and investigate the basic properties of the space. Also, we consider several inclusions under some conditions. Moreover, we discuss compact embeddings with other suitable spaces under some conditions and reveal multipliers of the space $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$. One of the our purpose of this paper is to generalize some of the results in [1] and [36] to the double weighted case.

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2. Notation and Preliminaries

Throughout this paper, we will work on G with Lebesgue measure dx. We denote by $C_c(G)$ as the linear space of continuous functions on G, which have compact support. The translation and character operators T_y and M_t are defined by $T_y f(x) = f(x-y)$ and $M_t f(y) = \langle y,t\rangle .f(y)$ respectively for $x,y \in G$, $t \in G$. Also $(B, \|.\|_B)$ is strongly translation invariant if one has $T_y B \subseteq B$ and $\|T_y f\|_B = \|f\|_B$ and strongly character invariant if $M_t B \subseteq B$ and $\|M_t f\|_B = \|f\|_B$ for all $f \in B$, $y \in G$ and $t \in G$.

A measurable and locally integrable function $\vartheta : G \longrightarrow (0,\infty)$ is called a weight function. Moreover the weight ϑ will be called a Beurling's weight function if $\vartheta(x) \geq 1$ and $\vartheta(x+y) \leq \vartheta(x)\vartheta(y)$ for all $x,y \in G$. We say that $\vartheta_1 \prec \vartheta_2$ if and only if there exists a $C>0$ such that $\vartheta_1(x) \leq C\vartheta_2(x)$ for all $x \in G$. Also, if $\vartheta_1 \prec \vartheta_2$ and $\vartheta_2 \prec \vartheta_1$ are satisfied, then we say that weight functions ϑ_1 and ϑ_2 are equivalent and denoted by $\vartheta_1 \approx \vartheta_2$. A weight function ϑ is said to satisfy the Beurling-Domar (shortly BD) condition if

$$\sum_{n\geq 1} \frac{\log \vartheta(nx)}{n^2} < \infty$$

for all $x \in G$, see [10]. Moreover, it is clear that every weight function is equivalent to a continuous weight, see [28, Lemma 4]. Hence, we deduce that $\vartheta(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. For example, if we choose the polynomial type weight function $\vartheta_s(x) = (1+|x|)^s$ for $s \geq 0$, then we have $\vartheta_s(x) \longrightarrow \infty$ as $x \longrightarrow \infty$, see [30].

For $1 \leq p < \infty$, the weighted Lebesgue space $L^p_\vartheta(G) = \{f : f\vartheta \in L^p(G)\}$ is a Banach space with norm $\|f\|_{p,\vartheta} = \|f\vartheta\|_p$ and its dual space $L^q_{\vartheta^{-1}}(G)$, where $\frac{1}{p}+\frac{1}{q}=1$. It is known that the space $L^p_\vartheta(G)$ is a reflexive Banach space for $1<p<\infty$. Moreover, for $p=1$, $L^1_\vartheta(G)$ is a Banach algebra under convolution, called a Beurling algebra. It is obvious that $\|.\|_1 \leq \|.\|_{1,\vartheta}$ and $L^1_\vartheta(G) \subset L^1(G)$. It is known that $L^p_{\vartheta_2}(G) \subset L^p_{\vartheta_1}(G)$ if and only if $\vartheta_1 \prec \vartheta_2$, see [12].

Now, assume that A is a Banach algebra. It is known that a Banach space B is called a Banach A-module if there exists a bilinear operation $\cdot : A \times B \longrightarrow B$ such that

(i) $(f\cdot g)\cdot h = f\cdot(g\cdot h)$ for all $f,g \in A$, $h \in B$.
(ii) For some constant $C \geq 1$, $\|f\cdot h\|_B \leq C\|f\|_A \|h\|_B$ for all $f \in A$, $h \in B$, see [11].

Moreover, $L^p_{loc}(G)$ is the space of all functions on G such that f restricted to any compact subset K of G belongs to $L^p(G)$. For $p=1$, the space $L^1_{loc}(G)$ is to be space of all measurable functions f on G such that $f.\chi_K \in L^1(G)$ for any compact subset $K \subset G$. Moreover, a Banach function space (shortly BF-space) on G is a Banach space $(B, \|.\|_B)$ of measurable functions which is continuously embedded into $L^1_{loc}(G)$, i.e. for any compact subset $K \subset G$ there exists some constant $C_K > 0$ such that $\|f.\chi_K\|_{L^1} \leq C_K.\|f\|_B$ for all $f \in B$. Also, a BF-space is called solid if $g \in B$, $f \in L^1_{loc}(G)$ and $|f(x)| \leq |g(x)|$ locally almost everywhere (shortly l.a.e) implies $f \in B$ and $\|f\|_B \leq \|g\|_B$. It is easy to see that $(B, \|.\|_B)$ is solid if and only if it is a L^∞-module. Let f be a measurable function on G.

Suppose that V and W are two Banach modules over a Banach algebra A. Then a multiplier from V into W is a bounded linear operator $T : V \longrightarrow W$, which commutes with module multiplication, i.e. $T(av) = aT(v)$ for $a \in A$ and $v \in V$. Also, we denote by $Hom_A(V,W)$ as the space of all multipliers from V into W. For convenience, we write that $Hom_A(V,V) = Hom_A(V)$. It is known that

$$Hom_A(V,W^*) \cong (V \otimes_A W)^*$$

where W^* is dual of W and $V \otimes_A W$ is the A-module tensor product of V and W,see [31, Corollary 2.13].

Moreover, the space $M(G)$ denotes all bounded regular Borel measures on G. Now, we define

$$M(\vartheta) = \left\{ \mu \in M(G) : \int_G \vartheta d|\mu| < \infty \right\}.$$

It is known that the space of multipliers from $L^1_\vartheta(G)$ to $L^1_\vartheta(G)$ is homeomorphic to the space $M(\vartheta)$,see [15].

In [7], Cigler revealed a generalization of Segal algebra. To define this we suppose that $S_\vartheta(G) = S_\vartheta$ is a subalgebra of $L^1_\vartheta(G)$ satisfying the conditions below.

(S1) S_ϑ is dense in $L^1_\vartheta(G)$.
(S2) S_ϑ is a Banach algebra under some norm $\|.\|_{S_\vartheta}$ and invariant under translations.
(S3) $\|T_y f\|_{S_\vartheta} \leq \vartheta(y) \|f\|_{S_\vartheta}$ for all $y \in G$ and for each $f \in S_\vartheta$.
(S4) If $f \in S_\vartheta$, then for every $\varepsilon > 0$ there is a neighborhood U of the identity element of G such that $\|T_y f - f\|_{S_\vartheta} < \varepsilon$ for all $y \in U$.
(S5) $\|f\|_{1,\vartheta} \leq \|f\|_{S_\vartheta}$ for all $f \in S_\vartheta$.

Denote the amalgam of L^p and ℓ^q on the real line is the normed space

$$(L^p, \ell^q) = \left\{ f \in L^p_{loc}(\mathbb{R}) : \|f\|_{pq} < \infty \right\}$$

equipped with the norm

$$\|f\|_{pq} = \left[\sum_{n=-\infty}^{\infty} \left[\int_n^{n+1} |f(x)|^p \, dx \right]^{\frac{q}{p}} \right]^{\frac{1}{q}}. \tag{1}$$

We make the appropriate changes for p, q infinite. The norm (1) makes the amalgam space (L^p, ℓ^q) into a Banach space, see [21]. Note that, the space $C_c(G)$ is a subspace of every amalgam spaces. Now, let $1 \leq p, q < \infty$. Then, the dual space of (L^p, ℓ^q) is isometrically isomorphic to $\left(L^{p'}, \ell^{q'}\right)$ where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$, see [1], [33].

Stewart [34] give an alternative definition of $(L^p, \ell^q)(G)$ based on the Structure Theorem [20, Theorem 24.30]. Indeed, let $G = \mathbb{R}^a \times G_1$, where a is a nonnegative integer and G_1 is a locally compact abelian group which contains an open compact subgroup H. Also, we denote $I = [0,1)^a \times H$ and $J = \mathbb{Z}^a \times T$ where T is a transversal of H in G_1, i.e. $G_1 = \bigcup_{t \in T} (t + H)$ is a coset decomposition of G_1. For $\alpha \in J$, we define $I_\alpha = \alpha + I$. Therefore G equals the disjoint union of relatively compact sets I_α. We normalize μ such that $\mu(I) = \mu(I_\alpha) = 1$ for all α. Let $1 \leq p, q \leq \infty$. The amalgam space $(L^p, \ell^q)(G) = (L^p, \ell^q)$ is a Banach space

$$\left\{ f \in L^p_{loc}(G) : \|f\|_{pq} < \infty \right\},$$

where

$$\begin{aligned} \|f\|_{pq} &= \left[\sum_{\alpha\in J} \|f\|^q_{L^p(I_\alpha)}\right]^{1/q} \text{ if } 1\le p,q<\infty, \qquad (2)\\ \|f\|_{\infty q} &= \left[\sum_{\alpha\in J} \sup_{x\in I_\alpha} |f(x)|^q\right]^{1/q} \text{ if } p=\infty,\ 1\le q<\infty,\\ \|f\|_{p\infty} &= \sup_{\alpha\in J} \|f\|_{L^p(I_\alpha)} \text{ if } 1\le p<\infty,\ q=\infty. \end{aligned}$$

If $G=\mathbb{R}$, then we have $J=\mathbb{Z}$, $I_\alpha=[\alpha,\alpha+1)$ and (2) becomes (1).

Throughout this paper, G will denote a locally compact abelian group with Haar measure and J and I_α will define as above. Moreover, we assume that $1\le p,q,r<\infty$ and every weights we used are Beurling's weight functions on G.

3. The Space $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$

Now, we define the vector space $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)=\left\{f\in\left(L^p_{\vartheta_1},\ell^1\right):\widehat{f}\in\left(L^q_{\vartheta_2},\ell^r\right)\right\}$ and equip with the norm

$$\|f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2}=\|f\|_{p1,\vartheta_1}+\left\|\widehat{f}\right\|_{qr,\vartheta_2}$$

for $f\in A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$. It is note that, since $\left(L^p_{\vartheta_1},\ell^1\right)$ is a subspace of $L^1(G)$, the Fourier transforms of the functions in $\left(L^p_{\vartheta_1},\ell^1\right)$ are well-defined.

The proof of the following theorem is clear, see [2].

Theorem 1. Let $1\le p,q<\infty$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\left(L^p_{\vartheta_1},\ell^q\right)$ and $\|f_n-f\|_{pq,\vartheta_1}\longrightarrow 0$ as $n\longrightarrow\infty$, where $f\in\left(L^p_{\vartheta_1},\ell^q\right)$. Then $(f_n)_{n\in\mathbb{N}}$ has a subsequence which converges pointwise almost everywhere to f.

Theorem 2. The space $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ is a Banach space with respect to $\|.\|^{p,1,q,r}_{\vartheta_1,\vartheta_2}$.

Proof. Assume that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$. Thus given $\varepsilon>0$, there is an $n_1\in\mathbb{N}$ such that for all $n,m\geqslant n_1$ implies

$$\begin{aligned} \|f_n-f_m\|^{p,1,q,r}_{\vartheta_1,\vartheta_2} &= \|f_n-f_m\|_{p1,\vartheta_1}+\left\|\widehat{f_n-f_m}\right\|_{qr,\vartheta_2}\\ &= \|f_n-f_m\|_{p1,\vartheta_1}+\left\|\widehat{f_n}-\widehat{f_m}\right\|_{qr,\vartheta_2}<\varepsilon. \end{aligned}$$

Therefore, $(f_n)_{n\in IN}\subset\left(L^p_{\vartheta_1},\ell^1\right)$ and $(f_n)_{n\in IN}\subset\left(L^q_{\vartheta_2},\ell^r\right)$ are Cauchy sequences with respect to $\|.\|_{p1,\vartheta_1}$ and $\|.\|_{qr,\vartheta_2}$, respectively. Since the spaces $\left(\left(L^p_{\vartheta_1},\ell^1\right),\|.\|_{p1,\vartheta_1}\right)$ and $\left(\left(L^q_{\vartheta_2},\ell^r\right),\|.\|_{qr,\vartheta_2}\right)$ are two Banach spaces, there exist $f\in\left(L^p_{\vartheta_1},\ell^1\right)$ and $g\in\left(L^q_{\vartheta_2},\ell^r\right)$ such that $\|f_n-f\|_{p1,\vartheta_1}\longrightarrow 0$, $\left\|\widehat{f_n}-g\right\|_{qr,\vartheta_2}\longrightarrow 0$. If we use the inequality $\|.\|_{1,\vartheta_1}\le\|.\|_{p1,\vartheta_1}$, then we get $\|f_n-f\|_{1,\vartheta_1}\longrightarrow 0$.

By Theorem 1, there is a subsequence $\left\{\widehat{f_{n_k}}\right\}_{k\in IN}$ of $\left\{\widehat{f_n}\right\}_{n\in IN}$ such that $\widehat{f_{n_k}} \longrightarrow g$ a.e. Since ϑ_1 is a Beurling's weight, we have

$$\left\|\widehat{f_n}-\widehat{f}\right\|_{\infty} \leq \left\|f_n-f\right\|_1 \leq \left\|f_n-f\right\|_{1,\vartheta_1}.$$

This follows that $\left\|\widehat{f_n}-\widehat{f}\right\|_{\infty} \longrightarrow 0$. Moreover, we get

$$\left\|\widehat{f_{n_k}}-\widehat{f}\right\|_{\infty} = \sup_k \left|f_{n_k}(x)-f(x)\right| \leq \left\|\widehat{f_n}-\widehat{f}\right\|_{\infty} \longrightarrow 0.$$

Therefore, we have $\widehat{f}=g$. This yields that

$$\left\|f_n-f\right\|_{\vartheta_1,\vartheta_2}^{p,1,q,r} = \left\|f_n-f\right\|_{p1,\vartheta_1} + \left\|\widehat{f_n}-\widehat{f}\right\|_{qr,\vartheta_2} \longrightarrow 0$$

and $f \in A_{\vartheta_1,\vartheta_2}^{p,1,q,r}(G)$. That is the desired result. ■

Theorem 3. If $1<p,q,r<\infty$, then the space $\left(A_{\vartheta_1,\vartheta_2}^{p,1,q,r}(G), \|.\|_{\vartheta_1,\vartheta_2}^{p,1,q,r}\right)$ is a Banach algebra with respect to convolution.

Proof. It is note that the space $\left(A_{\vartheta_1,\vartheta_2}^{p,1,q,r}(G), \|.\|_{\vartheta_1,\vartheta_2}^{p,1,q,r}\right)$ is a Banach space by the Theorem 2. Now, let $f,g \in A_{\vartheta_1,\vartheta_2}^{p,1,q,r}(G)$ be given. Thus, we have $f,g \in \left(L_{\vartheta_1}^p, \ell^1\right)$ and $\widehat{f},\widehat{g} \in \left(L_{\vartheta_2}^q, \ell^r\right)$. Since $\left(L_{\vartheta_1}^p, \ell^1\right)$ is a Banach algebra under convolution (see [33]), we have $f*g \in \left(L_{\vartheta_1}^p, \ell^1\right)$ and there exists $C\geq 1$ such that

$$\left\|f*g\right\|_{p1,\vartheta_1} \leq C\left\|f\right\|_{p1,\vartheta_1}\left\|g\right\|_{p1,\vartheta_1}. \tag{3}$$

If we consider the inequality

$$\left|\widehat{(f*g)}(x)\right| = \left|\widehat{f}(x)\right|\left|\widehat{g}(x)\right| \leq \left\|\widehat{f}\right\|_{\infty}\left|\widehat{g}(x)\right|,$$

then we have

$$\left\|\widehat{f*g}\right\|_{qr,\vartheta_2} \leq \left\|\widehat{f}\right\|_{\infty}\left\|\widehat{g}\right\|_{qr,\vartheta_2} \leq \left\|f\right\|_{p1,\vartheta_1}\left\|\widehat{g}\right\|_{qr,\vartheta_2} \tag{4}$$

and $\widehat{f*g} \in \left(L_{\vartheta_2}^q, \ell^r\right)$. Therefore, we have $f*g \in A_{\vartheta_1,\vartheta_2}^{p,1,q,r}(G)$. This follows by (3) and (4) that

$$\begin{aligned}\left\|f*g\right\|_{\vartheta_1,\vartheta_2}^{p,1,q,r} &= \left\|f*g\right\|_{p1,\vartheta_1} + \left\|\widehat{f*g}\right\|_{qr,\vartheta_2} \\ &\leq C\left\|f\right\|_{p1,\vartheta_1}\left(\left\|g\right\|_{p1,\vartheta_1} + \left\|\widehat{g}\right\|_{qr,\vartheta_2}\right) \\ &\leq C\left\|f\right\|_{\vartheta_1,\vartheta_2}^{p,1,q,r}\left\|g\right\|_{\vartheta_1,\vartheta_2}^{p,1,q,r}.\end{aligned}$$

■

Theorem 4. The space $\left(A_{\vartheta_1,\vartheta_2}^{p,1,q,r}(G), \|.\|_{\vartheta_1,\vartheta_2}^{p,1,q,r}\right)$ is a solid BF-space.

Proof. Assume that $K \subset G$ is a compact subset and let $f \in A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$. Since ϑ_1 is a Beurling's weight function and the space $L^p_{\vartheta_1}(G)$ is continuously embedded in $L^1_{\vartheta_1}(G)$, we have

$$\begin{aligned}\int_K |f(x)|\,dx &\leq \|f\|_1 \leq \|f\|_{1,\vartheta_1} \\ &\leq \|f\|_{p1,\vartheta_1} \leq \|f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2}.\end{aligned}$$

Let $f \in A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ and $g \in L^\infty(G)$. Since $\left(L^p_{\vartheta_1}, \ell^1\right)$ and $\left(L^q_{\vartheta_2}, \ell^r\right)$ are solid BF-space (see [3], [33]), then

$$\begin{aligned}\|fg\|^{p,1,q,r}_{\vartheta_1,\vartheta_2} &\leq \|f\|_{p1,\vartheta_1}\|g\|_\infty + \|f\|_{qr,\vartheta_2}\|g\|_\infty \\ &= \|f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2}\|g\|_\infty.\end{aligned}$$

This completes the proof. ∎

Theorem 5. The following statements hold.

(i) The space $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ is translation invariant and for every $f \in A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ and $y \in G$ the inequality $C_1(f)\,\vartheta_1(y) \leq \|T_y f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2} \leq C_2(f)\,\vartheta_1(y)$ holds where $C_1(f) > 0$ and $C_2(f) = \|f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2}$.

(ii) The map $y \longrightarrow T_y f$ is continuous from G into $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ for every $f \in A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$.

Proof. Let $f \in A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$. Since ϑ_1 is a Beurling's weight function, it is easy to see that $T_y f \in \left(L^p_{\vartheta_1}, \ell^1\right)$ and $\|T_y f\|_{p1,\vartheta_1} \leq \vartheta_1(y)\|f\|_{p1,\vartheta_1}$ for all $y \in G$.

Moreover, for $p > 1$, it is clear that $\left(L^p_{\vartheta_1}, \ell^1\right)$ is continuously embedded in $L^p_{\vartheta_1}(G)$, see [33]. Thus, if we consider the Lemma 2.2 in [12], then there is a constant $C > 0$ depending on f such that

$$C\vartheta_1(y) \leq \|T_y f\|_{p,\vartheta_1} \leq C^*\|T_y f\|_{p1,\vartheta_1} \leq C^*\|T_y f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2}$$

for all $y \in G$. This follows that

$$C_1\vartheta_1(y) \leq \|T_y f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2} \tag{5}$$

where $C_1 = \frac{C}{C^*}$ depends on f. It is clear that $\widehat{L_y f} = M_{-y}\widehat{f}$. Moreover, if we consider that the weighted amalgam space $\left(L^q_{\vartheta_2}, \ell^r\right)$ is strongly character invariant and the function $t \longrightarrow M_t f$ is continuous from $\widehat{G}$ into $\left(L^q_{\vartheta_2}, \ell^r\right)$ (see [3], [29], [32]), then we have

$$\left\|\widehat{T_y f}\right\|_{qr,\vartheta_2} = \left\|M_{-y}\widehat{f}\right\|_{qr,\vartheta_2} = \left\|\widehat{f}\right\|_{qr,\vartheta_2} < \infty.$$

This follows that $T_y f \in A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$. Since $\vartheta_1 \geq 1$, we get

$$\begin{aligned}\|T_y f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2} &\leq \vartheta_1(y)\|f\|_{p1,\vartheta_1} + \left\|\widehat{f}\right\|_{qr,\vartheta_2} \\ &\leq \vartheta_1(y)\|f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2}\end{aligned} \tag{6}$$

for all $y \in G$. This completes *(i)* by (5) and (6). It is obvious that T_y is linear. For any $\varepsilon > 0$, there is a neighbourhood V_1 of the unit element of G such that

$$\|T_y f - f\|_{p1,\vartheta_1} < \frac{\varepsilon}{2} \tag{7}$$

for all $y \in V_1$. Also, there is a neighbourhood V_2 of the unit element of G such that

$$\left\|\widehat{T_y f} - \widehat{f}\right\|_{qr,\vartheta_2} = \left\|M_{-y}\widehat{f} - \widehat{f}\right\|_{qr,\vartheta_2} < \frac{\varepsilon}{2} \tag{8}$$

for all $y \in V_2$. Now, let us denote $U = V_1 \cap V_2$. By (7) and (8), we have

$$\|T_y f - f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2} = \|T_y f - f\|_{p1,\vartheta_1} + \left\|\widehat{T_y f - f}\right\|_{qr,\vartheta_2} < \varepsilon$$

for all $y \in U$. That is the desired result. ∎

Theorem 6. Assume that ϑ_1 satisfies (BD) condition. Then the space $A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ is a S_{ϑ_1} algebra.

Proof. We have already proved the several conditions for S_{ϑ_1} algebra in Theorem 3 and Theorem 5. Now, let us denote

$$F_{0,\vartheta_1} = \left\{ f \in L^1_{\vartheta_1}(G) : \widehat{f} \text{ has a compact support} \right\}.$$

Since ϑ_1 satisfies (BD) condition, the set F_{0,ϑ_1} is dense in $L^1_{\vartheta_1}(G)$. It is clear that $C_c(G) \subset \left(L^q_{\vartheta_2}, \ell^r\right)$. Because of the fact that the inclusions $F_{0,\vartheta_1} \subset A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G) \subset L^1_{\vartheta_1}(G)$ hold and F_{0,ϑ_1} is dense in $L^1_{\vartheta_1}(G)$, then $A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ is dense in $L^1_{\vartheta_1}(G)$. That is the desired result. ∎

Theorem 7. Let $1 < p, q, r < \infty$. If $\vartheta_1 \prec \vartheta_0$, then $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ is a Banach $L^1_{\vartheta_0}(G)$-module with respect to $\|.\|^{p,1,q,r}_{\vartheta_1,\vartheta_2}$.

Proof. Since $\vartheta_1 \prec \vartheta_0$, we have $L^1_{\vartheta_0}(G) \hookrightarrow L^1_{\vartheta_1}(G)$. Moreover, if we consider the fact that $\left(L^p_{\vartheta_1}, \ell^1\right)$ is a Banach $L^1_{\vartheta_1}(G)$-module for $1 < p < \infty$, then there exists $C > 0$ such that

$$\begin{aligned}
\|f * g\|^{p,1,q,r}_{\vartheta_1,\vartheta_2} &\leq C\|f\|_{p1,\vartheta_1}\|g\|_{1,\vartheta_1} + \left\|\widehat{f}\right\|_{qr,\vartheta_2}\|\widehat{g}\|_\infty \\
&\leq C\|f\|_{p1,\vartheta_1}\|g\|_{1,\vartheta_0} + \left\|\widehat{f}\right\|_{qr,\vartheta_2}\|\widehat{g}\|_\infty \\
&\leq C\|f\|_{p1,\vartheta_1}\|g\|_{1,\vartheta_0} + \left\|\widehat{f}\right\|_{qr,\vartheta_2}\|g\|_{1,\vartheta_0} \\
&\leq \max\{1, C\}\|f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2}\|g\|_{1,\vartheta_0}
\end{aligned}$$

for any $f \in A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ and $g \in L^1_{\vartheta_0}(G)$. If we define a new norm $|||.|||$ on $L^1_{\vartheta_0}(G)$ such that $|||.||| = \max\{c_1, c_2\}\|.\|_{1,\vartheta_0}$, then this norm is equivalent to the norm $\|.\|_{1,\vartheta_0}$ on $L^1_{\vartheta_0}(G)$. This completes the proof. ∎

Theorem 8. Suppose that ϑ_1 satisfies (BD) condition. Then $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ has an approximate unit with compactly supported Fourier transforms.

Proof. Let $f \in A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ and $\varepsilon > 0$ be given. Then by Theorem 5, there exists a neighbourhood U of the unit element of G such that

$$\|T_y f - f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2} < \frac{\varepsilon}{2} \tag{9}$$

for all $y \in U$. Let be taken a non-negative function $g \in C_c(G)$ such that supp$g \subset U$ and $\int\limits_G g(y)dy = 1$. Since

$$g * f - f = \int\limits_G g(y)\left(T_y f - f\right) dy$$

by using the inequality (9), we get

$$\begin{aligned}
\|g * f - f\|_{\vartheta_1,\vartheta_2}^{p,1,q,r} &= \left\| \int\limits_G g(y)\left(T_y f - f\right) dy \right\|_{\vartheta_1,\vartheta_2}^{p,1,q,r} \\
&\leq \int\limits_G g(y) \left\|T_y f - f\right\|_{\vartheta_1,\vartheta_2}^{p,1,q,r} dy \\
&= \int\limits_U g(y) \left\|T_y f - f\right\|_{\vartheta_1,\vartheta_2}^{p,1,q,r} dy < \frac{\varepsilon}{2} \int\limits_G g(y)dy = \frac{\varepsilon}{2}. \qquad (10)
\end{aligned}$$

Moreover, since the set F_{0,ϑ_1} is dense in $L^1_{\vartheta_1}(G)$ by Theorem 6, there exists $h \in F_{0,\vartheta_1}$ such that

$$\|g - h\|_{1,\vartheta_1} < \frac{\varepsilon}{2\,\|f\|_{\vartheta_1,\vartheta_2}^{p,1,q,r}}. \qquad (11)$$

If we consider that the space $C_c(G)$ is included in all amalgam spaces, then it we have $h \in A_{\vartheta_1,\vartheta_2}^{p,1,q,r}(G)$. Also, it is clear that

$$\|(h-g) * f\|_{\vartheta_1,\vartheta_2}^{p,1,q,r} \leq \|(h-g)\|_{1,\vartheta_1} \|f\|_{\vartheta_1,\vartheta_2}^{p,1,q,r}. \qquad (12)$$

By (10), (11) and (12), we obtain

$$\begin{aligned}
\|h * f - f\|_{\vartheta_1,\vartheta_2}^{p,1,q,r} &\leq \|h * f - g * f\|_{\vartheta_1,\vartheta_2}^{p,1,q,r} + \|g * f - f\|_{\vartheta_1,\vartheta_2}^{p,1,q,r} \\
&< \frac{\varepsilon}{2\,\|f\|_{\vartheta_1,\vartheta_2}^{p,1,q,r}} \|f\|_{\vartheta_1,\vartheta_2}^{p,1,q,r} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

■

Consider the mapping Φ from $A_{\vartheta_1,\vartheta_2}^{p,1,q,r}(G)$ into $\left(L^p_{\vartheta_1}, \ell^1\right) \times \left(L^q_{\vartheta_2}, \ell^r\right)$ defined by $\Phi(f) = \left(f, \widehat{f}\right)$. This is a linear isometry from $A_{\vartheta_1,\vartheta_2}^{p,1,q,r}(G)$ into $\left(L^p_{\vartheta_1}, \ell^1\right) \times \left(L^q_{\vartheta_2}, \ell^r\right)$ in sense to the norm

$$\left\| \left(f, \widehat{f}\right) \right\| = \|f\|_{p1,\vartheta_1} + \left\|\widehat{f}\right\|_{qr,\vartheta_2}$$

for $f \in A_{\vartheta_1,\vartheta_2}^{p,1,q,r}(G)$. Hence it is easy to see that $A_{\vartheta_1,\vartheta_2}^{p,1,q,r}(G)$ is a closed subspace of the Banach space $\left(L^p_{\vartheta_1}, \ell^1\right) \times \left(L^q_{\vartheta_2}, \ell^r\right)$. Let

$$H = \left\{ \left(f, \widehat{f}\right) : f \in A_{\vartheta_1,\vartheta_2}^{p,1,q,r}(G) \right\}$$

and

$$K = \left\{ \begin{array}{c} (\varphi, \psi) : (\varphi, \psi) \in \left(L^{p'}_{\vartheta_1^{-1}}, \ell^\infty\right) \times \left(L^{q'}_{\vartheta_2^{-1}}, \ell^{r'}\right), \\ \int\limits_G f(x)\varphi(x)dx + \int\limits_G \widehat{f}(y)\psi(y)dy = 0, \text{ for all } \left(f, \widehat{f}\right) \in H \end{array} \right\},$$

where $\frac{1}{r} + \frac{1}{r'} = 1$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

The following theorem is easily proved by Duality Theorem 1.7 in [25].

Theorem 9. The dual space $\left(A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)\right)^*$ of $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ is isomorphic to $\left(L^{p'}_{\vartheta_1^{-1}},\ell^{\infty}\right)\times\left(L^{q'}_{\vartheta_2^{-1}},\ell^{r'}\right)/K$.

4. Inclusions of the spaces $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$

Theorem 10. The inclusion $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)\subset A^{p,1,q,r}_{\vartheta_3,\vartheta_4}(G)$ holds if and only if the space $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ is continuously embedded in $A^{p,1,q,r}_{\vartheta_3,\vartheta_4}(G)$.

Proof. The sufficient condition of the theorem is clear by definition of embedding. Now, assume that $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)\subset A^{p,1,q,r}_{\vartheta_3,\vartheta_4}(G)$ holds. Moreover, we define the sum norm $|||.|||=\|.\|^{p,1,q,r}_{\vartheta_1,\vartheta_2}+\|.\|^{p,1,q,r}_{\vartheta_3,\vartheta_4}$. It is easy to see that $\left(A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G),|||.|||\right)$ is a Banach space. Now, let us define the unit function I from $\left(A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G),|||.|||\right)$ into $\left(A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G),\|.\|^{p,1,q,r}_{\vartheta_1,\vartheta_2}\right)$. Since the inequality

$$\|I(f)\|^{p,1,q,r}_{\vartheta_1,\vartheta_2}=\|f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2}\le|||f|||$$

is satisfied, I is continuous. If we consider the Banach's theorem, then I is a homeomorphism, see [6]. That means the norms $|||.|||$ and $\|.\|^{p,1,q,r}_{\vartheta_1,\vartheta_2}$ are equivalent. Thus, for every $f\in A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ there exists $c>0$ such that

$$|||f|||\le c\,\|f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2}. \tag{13}$$

Therefore, by using (13) and the definition of norm $|||.|||$, we obtain

$$\|f\|^{p,1,q,r}_{\vartheta_3,\vartheta_4}\le|||f|||\le c\,\|f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2}.$$

That is the desired result. ∎

Now, we give some continuous embeddings of $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ under some conditions.

Theorem 11. The following statements are true.

(i) If $p_2\le p_1$ and $r_1\le r_2$, then we have $A^{p_1,1,q,r_1}_{\vartheta_1,\vartheta_2}(G)\hookrightarrow A^{p_2,1,q,r_2}_{\vartheta_1,\vartheta_2}(G)$.

(ii) If $\vartheta_3\prec\vartheta_1$ and $\vartheta_4\prec\vartheta_2$, then $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ is continuously embedded in $A^{p,1,q,r}_{\vartheta_3,\vartheta_4}(G)$.

(iii) If $p_2\le p_1$ and $q_2\le q_1$, then we get $A^{p_1,1,q_1,r}_{\vartheta_1,\vartheta_2}(G)\hookrightarrow A^{p_2,1,q_2,r}_{\vartheta_1,\vartheta_2}(G)$.

(iv) If $\vartheta_3\prec\vartheta_1$, $q_2\le q_1$ and $r_1\le r_2$, then the space $A^{p,1,q_1,r_1}_{\vartheta_1,\vartheta_2}(G)$ is continuously embedded in $A^{p,1,q_2,r_2}_{\vartheta_3,\vartheta_2}(G)$.

Proof. Let $p_2\le p_1$ and $r_1\le r_2$. Moreover, since the embeddings $\left(L^{p_1}_{\vartheta_1},\ell^1\right)\hookrightarrow\left(L^{p_2}_{\vartheta_1},\ell^1\right)$ and $\left(L^{q}_{\vartheta_2},\ell^{r_1}\right)\hookrightarrow\left(L^{q}_{\vartheta_2},\ell^{r_2}\right)$ hold under these hypotheses (see [19], [33]), there exist $c_1,c_2>0$ such that

$$\begin{aligned}\|f\|^{p_2,1,q,r_2}_{\vartheta_1,\vartheta_2}&\le c_1\|f\|_{p_11,\vartheta_1}+c_2\left\|\widehat{f}\right\|_{qr_1,\vartheta_2}\\&\le\max\{c_1,c_2\}\|f\|^{p_1,1,q,r_1}_{\vartheta_1,\vartheta_2}\end{aligned}$$

for all $f\in A^{p_1,1,q,r_1}_{\vartheta_1,\vartheta_2}(G)$. This completes (i). Now, let $f\in A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ be given. Hence, we get $f\in\left(L^{p}_{\vartheta_1},\ell^1\right)$ and $\widehat{f}\in\left(L^{q}_{\vartheta_2},\ell^{r}\right)$. Moreover, assume that $\vartheta_3\prec\vartheta_1$ and $\vartheta_4\prec\vartheta_2$ hold. This follows that there exist $c_3, c_4 > 0$ such that $\vartheta_3(x) \le c_3\vartheta_1(x)$ and $\vartheta_4(x) \le c_4\vartheta_2(x)$ for all $x \in G$. Therefore, we have

$$\begin{aligned} \|f\|^{p,1,q,r}_{\vartheta_3,\vartheta_4} &= \sum_{n\in J} \|f\|_{L^p_{\vartheta_3}(I_n)} + \left(\sum_{m\in J} \left\|\widehat{f}\right\|^r_{L^q_{\vartheta_4}(I_m)}\right)^{\frac{1}{r}} \\ &\le c_3 \sum_{n\in J} \|f\|_{L^p_{\vartheta_1}(I_n)} + c_4 \left(\sum_{m\in J} \left\|\widehat{f}\right\|^r_{L^q_{\vartheta_2}(I_m)}\right)^{\frac{1}{r}} \\ &\le \max\{c_3, c_4\} \|f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2}. \end{aligned}$$

This proves (ii). If we consider (i) and (ii), then we obtain (iii) and (iv). ■

Theorem 12. Let ϑ, ϑ_1 and ϑ_2 be Beurling's weights. Then the space $A^{p,1,q,r}_{\vartheta_1,\vartheta}(G)$ is continuously embedded in $A^{p,1,q,r}_{\vartheta_2,\vartheta}(G)$ if and only if $\vartheta_2 \prec \vartheta_1$.

Proof. Assume that $A^{p,1,q,r}_{\vartheta_1,\vartheta}(G) \hookrightarrow A^{p,1,q,r}_{\vartheta_2,\vartheta}(G)$. By the Theorem 5, there are $C_1, C_2, C_3, C_4 > 0$ such that

$$C_1\vartheta_1(y) \le \|T_y f\|^{p,1,q,r}_{\vartheta_1,\vartheta} \le C_2\vartheta_1(y) \tag{14}$$

and

$$C_3\vartheta_2(y) \le \|T_y f\|^{p,1,q,r}_{\vartheta_2,\vartheta} \le C_4\vartheta_2(y) \tag{15}$$

for $y \in G$. Since $T_y f \in A^{p,1,q,r}_{\vartheta_1,\vartheta}(G)$ for every $f \in A^{p,1,q,r}_{\vartheta_1,\vartheta}(G)$, there is a $C > 0$ such that

$$\|T_y f\|^{p,1,q,r}_{\vartheta_2,\vartheta} \le C \|T_y f\|^{p,1,q,r}_{\vartheta_1,\vartheta}. \tag{16}$$

If we consider the (14), (15) and (16), then we have

$$C_3\vartheta_2(y) \le \|T_y f\|^{p,1,q,r}_{\vartheta_2,\vartheta} \le C \|T_y f\|^{p,1,q,r}_{\vartheta_1,\vartheta} \le CC_2\vartheta_1(y).$$

This follows that $\vartheta_2 \prec \vartheta_1$. The rest of the proof can be proven by the similar in Theorem 11. ■

The following corollary can be easily proven by using the Theorem 11 and Theorem 12.

Corollary 1. The following expressions are true.

(i) The equality $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G) = A^{p,1,q,r}_{\vartheta_3,\vartheta_4}(G)$ is satisfied if $\vartheta_1 \approx \vartheta_3$ and $\vartheta_2 \approx \vartheta_4$.

(ii) If $p_2 \le p_1$, $q_2 \le q_1$ and $r_1 \le r_2$, then we get $A^{p_1,1,q_1,r_1}_{\vartheta_1,\vartheta_2}(G) \hookrightarrow A^{p_1,1,q_2,r_2}_{\vartheta_1,\vartheta_2}(G)$.

(iii) Let $p_2 \le p_1$, $q_2 \le q_1$ and $r_1 \le r_2$. If $\vartheta_3 \prec \vartheta_1$ and $\vartheta_4 \prec \vartheta_2$, then we have $A^{p_1,1,q_1,r_1}_{\vartheta_1,\vartheta_2}(G) \hookrightarrow A^{p_1,1,q_2,r_2}_{\vartheta_3,\vartheta_4}(G)$.

(iv) The expression $A^{p,1,q,r}_{\vartheta_1,\vartheta}(G) = A^{p,1,q,r}_{\vartheta_2,\vartheta}(G)$ holds if and only if $\vartheta_2 \approx \vartheta_1$.

5. Compact Embeddings of the Space $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(\mathbb{R}^d)$

Now, we investigate compact embeddings of the spaces $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ with the similar methods in [18]. Also, we will take $G = \mathbb{R}^d$ with Lebesgue measure dx for compact embeddings.

Theorem 13. Let $1 < p, q, r < \infty$. Assume that $(f_n)_{n\in\mathbb{N}}$ is a sequence in $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$. If $(f_n)_{n\in\mathbb{N}}$ converges to zero in $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$, then $(f_n)_{n\in\mathbb{N}}$ converges to zero in the vague topology (which means that

$$\int\limits_{\mathbb{R}^d} f_n(x)k(x)dx \longrightarrow 0$$

for $n \longrightarrow \infty$ for all $k \in C_c(\mathbb{R}^d)$,see [8]).

Proof. Let $k \in C_c(\mathbb{R}^d)$. For $p > 1$, since the space $\left(L^p_{\vartheta_1}, \ell^1\right)$ is continuously embedded in $L^p_{\vartheta_1}\left(\mathbb{R}^d\right)$, we have

$$\begin{aligned} \left|\int\limits_{\mathbb{R}^d} f_n(x)k(x)dx\right| &\leq \|k\|_{p',\vartheta_1}\|f_n\|_{p,\vartheta_1} \leq C\|k\|_{p',\vartheta_1}\|f_n\|_{p1,\vartheta_1} \\ &\leq \|k\|_{p',\vartheta_1}\|f_n\|^{p,1,q,r}_{\vartheta_1,\vartheta_2} \end{aligned}$$

where $\frac{1}{p'} + \frac{1}{p} = 1$ by the Hölder inequality. Therefore, the sequence $(f_n)_{n\in\mathbb{N}}$ converges to zero in vague topology. ■

Theorem 14. If $\vartheta \prec \vartheta_1$ and $\frac{\vartheta(x)}{\vartheta_1(x)}$ doesn't tend to zero in $\mathbb{R}^d$ as $x \longrightarrow \infty$, then the embedding of the space $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$ into $\left(L^p_{\vartheta}, \ell^1\right)$ is never compact.

Proof. Since $\vartheta \prec \vartheta_1$, there exists $C_1 > 0$ such that $\vartheta(x) \leq C_1\vartheta_1(x)$ for all $x \in \mathbb{R}^d$. This follows that $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right) \subset \left(L^p_{\vartheta}, \ell^1\right)$. Assume that $(t_n)_{n\in\mathbb{N}}$ is a sequence with $t_n \longrightarrow \infty$ in $\mathbb{R}^d$. Since $\frac{\vartheta(x)}{\vartheta_1(x)}$ does not tend to zero as $x \longrightarrow \infty$, there exists $\delta > 0$ such that $\frac{\vartheta(x)}{\vartheta_1(x)} \geq \delta > 0$ for $x \longrightarrow \infty$. To end the proof, we take any fixed $f \in A^{p,1,q,r}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$ and define a sequence of functions $(f_n)_{n\in\mathbb{N}}$ where $f_n = (\vartheta_1(t_n))^{-1} T_{t_n} f$. This sequence is bounded in $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$. Indeed, by Theorem 5, we get

$$\begin{aligned} \|f_n\|^{p,1,q,r}_{\vartheta_1,\vartheta_2} &= (\vartheta_1(t_n))^{-1}\|T_{t_n}f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2} \\ &\leq (\vartheta_1(t_n))^{-1}\vartheta_1(t_n)\|f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2} = \|f\|^{p,1,q,r}_{\vartheta_1,\vartheta_2} \end{aligned}$$

Now, we will prove that there would not exists norm convergence of subsequence of $(f_n)_{n\in\mathbb{N}}$ in $\left(L^p_{\vartheta}, \ell^1\right)$. The sequence mentioned above converges to zero in sense to the vague topology. To prove this, for every $k \in C_c(\mathbb{R}^d)$ we obtain

$$\begin{aligned} \left|\int\limits_{\mathbb{R}^d} f_n(x)k(x)dx\right| &\leq \frac{1}{\vartheta_1(t_n)}\|k\|_{p',\vartheta_1}\|f_n\|_{p,\vartheta_1} \\ &\leq \frac{C_1}{\vartheta_1(t_n)}\|k\|_{p',\vartheta_1}\|f_n\|^{p,1,q,r}_{\vartheta_1,\vartheta_2} \qquad (17) \end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $C_1 > 0$. Since the right side of (17) tends zero for $n \longrightarrow \infty$, we have

$$\int\limits_{\mathbb{R}^d} f_n(x)k(x)dx \longrightarrow 0.$$

If we consider the Theorem 13, the only possible limit of $(f_n)_{n\in\mathbb{N}}$ in $\left(L^p_\vartheta, \ell^1\right)$ is zero. This follows by Lemma 2.2 in [12] that there exist $C_2, C_3 > 0$ depending on f such that

$$C_2\vartheta(y) \leq \|T_y f\|_{p,\vartheta} \leq C_3\vartheta(y)$$

for all $y \in \mathbb{R}^d$. Thus we have

$$\begin{aligned} \|f_n\|_{p1,\vartheta} &= (\vartheta_1(t_n))^{-1} \|T_{t_n} f\|_{p1,\vartheta} \geq \frac{1}{C_1} (\vartheta_1(t_n))^{-1} \|T_{t_n} f\|_{p,\vartheta} \\ &\geq \frac{C_2}{C_1} (\vartheta_1(t_n))^{-1} \vartheta(y). \end{aligned} \tag{18}$$

Since $\frac{\vartheta(t_n)}{\vartheta_1(t_n)} \geq \delta > 0$ for all t_n, by using (18) we get

$$\|f_n\|_{p1,\vartheta} \geq C (\vartheta_1(t_n))^{-1} \vartheta(y) \geq C\delta$$

where $C = \frac{C_2}{C_1}$. Thus there would not possible to find norm convergent subsequence of $(f_n)_{n\in\mathbb{N}}$ in $\left(L^p_\vartheta, \ell^1\right)$. This completes the proof. ■

Theorem 15. If $\vartheta_3 \prec \vartheta_1$ and $\frac{\vartheta_3(y)}{\vartheta_1(y)}$ doesn't tend to zero in $\mathbb{R}^d$, then the embedding $i : A^{p,1,q,r}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right) \hookrightarrow A^{p,1,q,r}_{\vartheta_3,\vartheta_2}\left(\mathbb{R}^d\right)$ is never compact.

Proof. Since $\vartheta_3 \prec \vartheta_1$, then it is clear that $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right) \subset A^{p,1,q,r}_{\vartheta_3,\vartheta_2}\left(\mathbb{R}^d\right)$. It is also known by Theorem 10 that the unit map from $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$ into $A^{p,1,q,r}_{\vartheta_3,\vartheta_2}\left(\mathbb{R}^d\right)$ is continuous. Now take any bounded sequence of $(f_n)_{n\in\mathbb{N}}$ in $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$. If there exists norm convergent subsequence of $(f_n)_{n\in\mathbb{N}}$ in $A^{p,1,q,r}_{\vartheta_3,\vartheta_2}\left(\mathbb{R}^d\right)$, this subsequence also converges in $\left(L^p_{\vartheta_3}, \ell^1\right)$. But this is a contradiction because of the fact that the embedding of the space $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$ into $\left(L^p_{\vartheta_3}, \ell^1\right)$ is never compact by the Theorem 14. ■

Note that if we define the sequence in Theorem 14 as $f_n = (\vartheta_2(t_n))^{-1} T_{t_n} f$, then the proof of the following theorem is similar with the previous one.

Theorem 16. If $\vartheta_1 \prec \vartheta_2, \vartheta_3 \prec \vartheta_2$ and $\frac{\vartheta_3(x)}{\vartheta_2(x)}$ does not tend to zero in $\mathbb{R}^d$ as $x \longrightarrow \infty$, then the embedding of the space $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$ into $\left(L^p_{\vartheta_3}, \ell^1\right)$ is never compact.

Theorem 17. The embedding of the space $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$ into $A^{p,1,q,r}_{\vartheta_3,\vartheta_4}\left(\mathbb{R}^d\right)$ is never compact if

(i) $\vartheta_4 \prec \vartheta_2 \prec \vartheta_1, \vartheta_3 \prec \vartheta_1$ and $\frac{\vartheta_3(x)}{\vartheta_1(x)}$ does not tend to zero in $\mathbb{R}^d$ as $x \longrightarrow \infty$, or

(ii) $\vartheta_3 \prec \vartheta_1 \prec \vartheta_2, \vartheta_4 \prec \vartheta_2$ and $\frac{\vartheta_3(x)}{\vartheta_2(x)}$ does not tend to zero in $\mathbb{R}^d$ as $x \longrightarrow \infty$.

Proof. Let $\vartheta_4 \prec \vartheta_2 \prec \vartheta_1, \vartheta_3 \prec \vartheta_1$. Thus, there exist $C_1, C_2 > 0$ such that $\vartheta_4(x) \leq C_1\vartheta_2(x)$ and $\vartheta_3(x) \leq C_2\vartheta_1(x)$ for all $x \in \mathbb{R}^d$. This follows that $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right) \subset A^{p,1,q,r}_{\vartheta_3,\vartheta_4}\left(\mathbb{R}^d\right)$ and the unit function $I : A^{p,1,q,r}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right) \longrightarrow A^{p,1,q,r}_{\vartheta_3,\vartheta_4}\left(\mathbb{R}^d\right)$ is continuous. Now, suppose that $\frac{\vartheta_3(x)}{\vartheta_1(x)}$ does not tend to zero in $\mathbb{R}^d$ as $x \longrightarrow \infty$ and $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$. If any subsequence of $(f_n)_{n\in\mathbb{N}}$ is convergent in $A^{p,1,q,r}_{\vartheta_3,\vartheta_4}\left(\mathbb{R}^d\right)$, then this subsequence is also convergent in $\left(L^p_{\vartheta_3}, \ell^1\right)$. However, this conflict with Theorem 14, because of the fact that the embedding of $A^{p,1,q,r}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$ into $\left(L^p_{\vartheta_3}, \ell^1\right)$ is never compact. This completes *(i)*. In similar way, *(ii)* can be proved. ■

6. Multipliers of $A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)$

In this section, we discuss multipliers of the spaces $A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)$. We define the space

$$M_{A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)} = \{\mu \in M(\vartheta_1) : \|\mu\|_M \leq C(\mu)\}$$

where

$$\|\mu\|_M = \sup\left\{ \frac{\|\mu * f\|^{1,1,q,r}_{\vartheta_1,\vartheta_2}}{\|f\|_{11,\vartheta_1}} : f \in \left(L^1_{\vartheta_1}, \ell^1\right),\ f \neq 0,\ \widehat{f} \in C_c(\widehat{G}) \right\}.$$

By [17, Proposition 2.1], we get $M_{A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)} \neq \{0\}$.

Theorem 18. If ϑ_1 satisfies (BD) condition, then for a linear operator $T : \left(L^1_{\vartheta_1}, \ell^1\right) \longrightarrow A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ the assertions below are equivalent:

(i) $T \in Hom_{\left(L^1_{\vartheta_1},\ell^1\right)}\left(\left(L^1_{\vartheta_1}, \ell^1\right), A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)\right)$.

(ii) There exists a unique $\mu \in M_{A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)}$ such that $Tf = \mu * f$ for every $f \in \left(L^1_{\vartheta_1}, \ell^1\right)$. Moreover the correspondence between T and μ defines an isomorphism between $Hom_{\left(L^1_{\vartheta_1},\ell^1\right)}\left(\left(L^1_{\vartheta_1}, \ell^1\right), A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)\right)$ and $M_{A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)}$.

Proof. It is known that $A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ is a S_{ϑ_1} space by Theorem 6. Thus, we get the desired result if we consider the Proposition 2.4 in [17]. ■

Theorem 19. If ϑ_1 satisfies (BD) condition and $T \in Hom_{\left(L^1_{\vartheta_1},\ell^1\right)}\left(A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)\right)$, then there is a unique pseudo measure $\sigma \in \left(A(\widehat{G})\right)^*$ (see [30]) such that $Tf = \sigma * f$ for all $f \in A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)$.

Proof. It is known that $A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)$ is a S_{ϑ_1} space by Theorem 6 and a Banach module over $\left(L^1_{\vartheta_1}, \ell^1\right)$ by Theorem 7. Thus, the proof is completed by Theorem 5 in [9]. ■

Theorem 20. The multiplier space $Hom_{\left(L^1_{\vartheta_1},\ell^1\right)}\left(\left(L^1_{\vartheta_1}, \ell^1\right), \left(A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)\right)^*\right)$ is isomorphic to $\left(L^\infty_{\vartheta_1^{-1}}, \ell^\infty\right) \times \left(L^{q'}_{\vartheta_2^{-1}}, \ell^{r'}\right)/K$.

Proof. By Theorem 7, we write $\left(L^1_{\vartheta_1}, \ell^1\right) * A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G) = A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)$. Hence by Corollary 2.13 in [31] and Theorem 9, we have

$$\begin{aligned} & Hom_{\left(L^1_{\vartheta_1},\ell^1\right)}\left(\left(L^1_{\vartheta_1}, \ell^1\right), \left(A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)\right)^*\right) \\ = & \left(\left(L^1_{\vartheta_1}, \ell^1\right) * A^{1,1,q,r}_{\vartheta_1,\vartheta_2}(G)\right)^* \\ = & \left(L^\infty_{\vartheta_1^{-1}}, \ell^\infty\right) \times \left(L^{q'}_{\vartheta_2^{-1}}, \ell^{r'}\right)/K \end{aligned}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$. ■

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